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P, S, *and* Q *refer to Proposals, Solutions and Quickies, respectively. Page numbers are given in parentheses. Thus,* P1212(111) *refers to proposal number* 1212 *which appears on page* 111.

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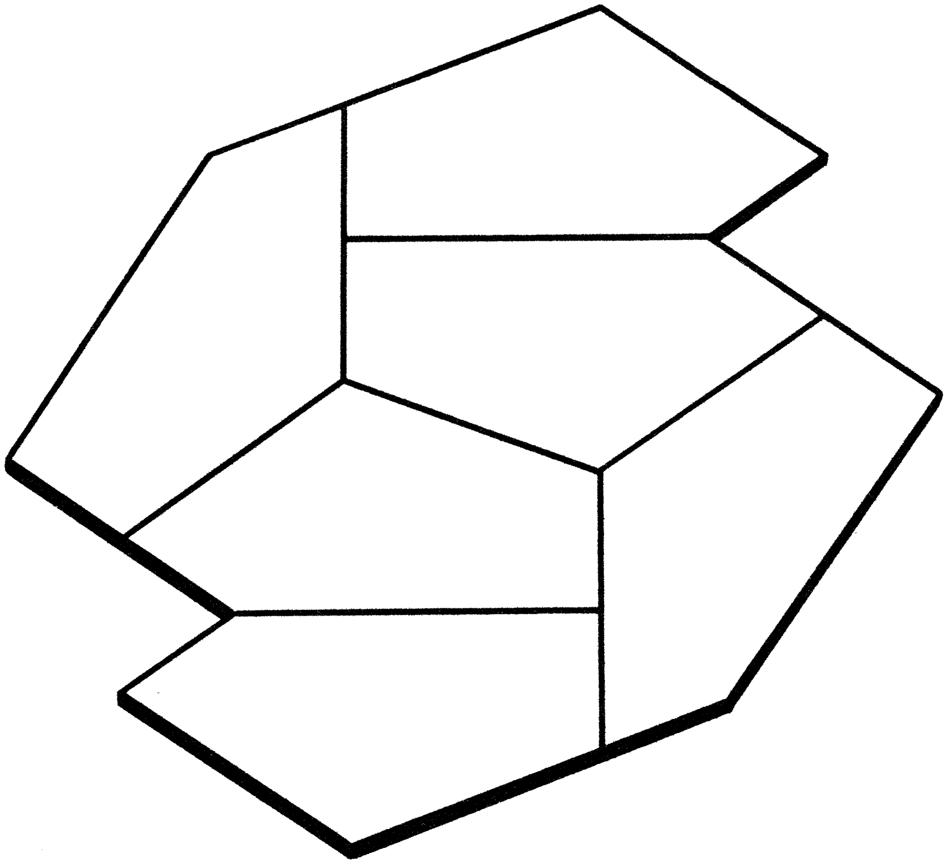

Vol. 58 No. 5
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BULGARIAN SOLITAIRE • FUZZY SKIING • BEST BET
CONSTRUCTIVE MATHEMATICS • INFINITE MATRICES

Volume 58, Number 5, November 1985
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MATHEMATICS MAGAZINE

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COVER: A unique new pentagon, discovered by Rolf Stein. See p. 308.

EDITORIAL POLICY

Mathematics Magazine is a journal which aims to provide inviting, informal mathematical exposition. Manuscripts for the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

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Send new manuscripts to: G. L. Alexanderson, Editor-elect, Mathematics Magazine, University of Santa Clara, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

AUTHORS

Kiyoshi Igusa ("Solution of the Bulgarian Solitaire Conjecture") calls himself a "differential topologist" (at least that's the subject area of his Ph.D. thesis from Princeton), but likes to dabble in other areas of mathematics as well. He wrote four papers in algebra with his wife, Gordana, and wasted a lot of time attacking such famous unsolved problems as the dense packing of spheres, the factorization of large numbers, and the $3k+1$ problem. He was really happy when he found a Martin Gardner problem in *Scientific American* that he could actually solve. He hasn't been able to win a single "Games" T-shirt—does *Scientific American* give T-shirts for solving its puzzles?

Mark Mandelkern ("Constructive Mathematics") worked in quite nonconstructive mathematics for a number of years, until 1970, when he first learned about constructive mathematics from Stanley Tennenbaum. Stanley, while not himself a constructivist, did much to inform others of Errett Bishop's innovative work, and he has made great efforts to promote a more open and free atmosphere in American mathematics; it is to him that the article "Constructive Mathematics" is dedicated.

Mandelkern's experience as truck driver, mariner, repairman, engineer, and college teacher has inclined him toward relevant mathematics. His recent research has been focused mainly on the constructivity of continuity theorems.

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ARTICLES

Solution of the Bulgarian Solitaire Conjecture

It's the gaps, not the stacks, that are the key to showing this game can always be won neatly.

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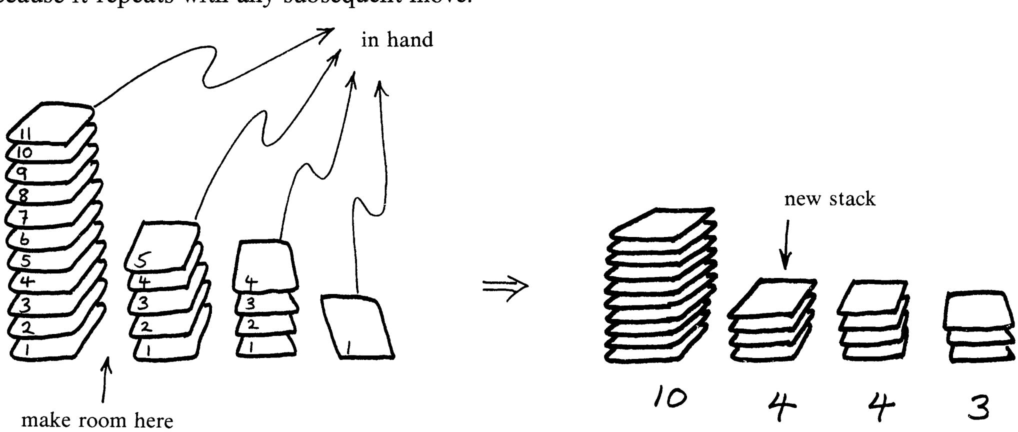
In July 1983 I was introduced to the game of Bulgarian solitaire in an article by Martin Gardner in *Scientific American* [**4**]. I was very intrigued by the game and I couldn't rest until I had resolved the conjecture associated with it. The starting point of my research was Jørgen Brandt's result [**3**], which is explained in detail in a recent paper of Ethan Akin and Morton Davis [**1**]. Brandt, Akin, and Davis analyzed all possible final outcomes of a game of Bulgarian solitaire. I, on the other hand, was more interested in how long it would take to reach that final outcome. In particular I wanted to solve the **Bulgarian solitaire conjecture**, which says that *a game of Bulgarian solitaire with a triangular number* $(k(k+1)/2 = 1+2+3+\cdots+k)$ *of cards can always be won in* $k(k-1)$ *moves or less*. This paper gives a detailed account of my proof of this conjecture.

I should start by explaining what the game is. The rules are very simple. You start with a deck of cards. It is not necessary to have exactly 52 cards. You can start with any finite number of cards, say, N.

The first thing you do is to divide the N cards into one or more stacks. For example, 21 cards could be divided into four stacks consisting of 11, 5, 4, and 1 cards. Now you can play the game. Take one card from each stack. Since it doesn't matter which card you take, you might as well take the top card. In the example you would get four cards. Put these cards down on the table to form a new stack. This gives you four stacks consisting of 10, 4, 4, and 3 cards. If you do this 24 more times you get six stacks consisting of 6, 5, 4, 3, 2, 1 cards. This is called the **winning position** because it repeats with any subsequent move.



In general, the game of Bulgarian solitaire is considered to be won if you get k stacks consisting of $k, k-1, k-2, \ldots, 3, 2, 1$ cards. This means you cannot win the game unless you start with $N = 1 + 2 + 3 + \cdots + k = k(k+1)/2$ for some k. These numbers are called **triangular numbers**. I use the notation $T_k = k(k+1)/2$.

In the example, $k = 6$ and $T_6 = 21$. The Bulgarian solitaire conjecture says that any game with 21 cards can be won in $k(k-1) = 30$ moves or less. We won in 25 moves; however the initial division of 21 cards into seven stacks with $5, 5, 4, 3, 2, 1, 1$ cards requires 30 moves.

To play Bulgarian solitaire you really don't need a deck of cards. All you need is pencil and paper. Instead of cutting a deck of cards you can write down a sequence of positive numbers, for example, $13, 9, 5$. This is called a **partition** of 27. It represents a way of dividing 27 objects into 3 groups. The individual numbers 13, 9, and 5 are called the **parts** of the partition. The partition has **size** 3 since there are 3 parts. To make one move of Bulgarian solitaire on this partition, you subtract 1 from each number (= part) and add to this new sequence the new part, 3. This gives you the new partition: $12, 8, 4, \underline{3}$. I underlined the number 3 to show that it is the new part which didn't occur in the previous partition. If you make one more move you get $11, 7, 3, 2, \underline{4}$ or $11, 7, \underline{4}, 3, 2$. It doesn't matter in which order you write the parts in a partition; I like to write them in decreasing order.

It is known that any game of Bulgarian solitaire can be won if you start with a partition of a triangular number. The first person to prove this was Jørgen Brandt [**2**]. Brandt also figured out what eventually happens to a game of Bulgarian solitaire with a nontriangular number. His findings can be phrased as follows: *Given any partition of any $N > 0$, a game of Bulgarian solitaire will eventually end in a cycle of partitions which keep repeating.* This is obvious because there are only a finite number of partitions of N. What is not obvious is *which* partitions repeat.

THEOREM 1. (Brandt [**3**]; see also [**1**], Theorem 4.)

(a) *If N is a triangular number, say, $N = T_k$, then the only repeating partition is the winning partition $k, k-1, k-2, \ldots, 2, 1$.*

(b) *If N is not triangular and T_k is the smallest triangular number larger than N ($T_{k-1} < N < T_k$), then the repeating partitions of N are all partitions $a_1, a_2, \ldots, a_n$ where $n = k$ or $k-1$ and each $a_i = k - i$ or $k - i + 1$. (In other words, a partition of N repeats if and only if each part is equal to or one less than the corresponding part of the winning partition of T_k and the size of the partition is equal to or one less than k, the size of the winning partition of T_k.)*

For example, if $N = 25$, then $k = 7$ because $T_7 = 28$ and $T_6 = 21$. The partition 11, 9, 5 of 25 becomes the partition $7, 6, 5, 4, 2, 1$ after 4 moves of Bulgarian solitaire. This partition repeats every 7 moves. You can see that this satisfies the conditions of Theorem 1(b) by writing it underneath the winning partition of T_7:

winning partition of T_7:	$7, 6, 5, 4, 3, 2, 1$
repeating partition of 25:	$7, 6, 5, 4, 2, 1.$

Brandt's results focused on how every game of Bulgarian solitaire ends in a cycle of repeating partitions, but did not answer the question of *how long* it would take to reach such a cycle. When $N = T_k$ it was conjectured that each game reaches a repeating (winning) partition in $k(k-1)$ moves or less. My proof of this conjecture is described in this article. The question of how long it takes to reach a repeating partition when N is not a triangular number is still not solved. The most I can say is that you always reach a repeating partition in less than $k(k-1)$ moves if $N < T_k$. This is not the best possible bound because, for example, I don't know of any partition of $N < T_k$ which requires exactly $k(k-1) - 1$ moves.

Gaps are the key

Although the details of the proof get complicated, the basic idea behind my solution of the Bulgarian solitaire conjecture is very simple. If you play a few games of Bulgarian solitaire you will find that there are certain patterns which occur only at the beginning of the game. For example, a partition of the form $x, x, x, x, 5, 2, \ldots$ or $x, x, x, x, 7, 3, \ldots$ can never occur after more than 4 moves of Bulgarian solitaire. I will explain why later. The point is that by looking at a small portion of a partition you can tell that the game has not progressed more than 4 moves.

By analyzing many examples I accumulated a list of patterns, which I call "gaps," so that the existence of a "gap" in any partition means that the game of Bulgarian solitaire has not progressed beyond a certain point. More precisely:

THEOREM 2. *After $h(h-1)$ moves of Bulgarian solitaire there are no "gaps" with "height" h or less. (In other words, the existence of a "gap" with "height" h means that the game has not progressed more than $h(h-1)-1$ moves.)*

After compiling a list of all possible "gaps," I found that every nonrepeating partition has at least one "gap." More precisely:

THEOREM 3. *If $N \leqslant T_k$ then every nonrepeating partition of N has at least one "gap" with "height" $h \leqslant k$.*

The Bulgarian solitaire conjecture follows easily from Theorems 2 and 3. If you have a partition of $N \leqslant T_k$, then after $k(k-1)$ moves Theorem 2 says there will be no "gaps" with "height" k or less. Theorem 3 says that this must be a repeating partition. This proves:

COROLLARY. *Every partition of $N \leq T_k = k(k+1)/2$ becomes a repeating partition after at most $k(k-1)$ moves of Bulgarian solitaire.*

Frank Bernhart pointed out to me that this generalization of the Bulgarian solitaire conjecture follows easily from the ordinary Bulgarian solitaire conjecture for $N = T_k$ and the comparison theorem (Theorem 3c of [**1**]). However, I don't want to restrict my discussion to triangular numbers.

The remainder of this article will be devoted to an explanation of what "gaps" are and how they change with each move of Bulgarian solitaire. I will try to explain in a step-by-step logical manner how very simple patterns which I call "stable gaps" give rise to "multiple gaps," "hidden gaps," and "external gaps." These are the key to the proof of Theorems 2 and 3.

In any sequence of numbers you will see patterns. In Bulgarian solitaire you will see moving patterns of numbers. For example, take the partition $12, 11, 10, 7, 4, 4, 4, 2, 1$ of $N = T_{10} = 55$. After one move this becomes $11, 10, \underline{9}, 9, 6, 3, 3, 3, 1$:

Partition #1:	12	11	10	7	4	4	4	2	1	
Partition #2:	11	10	$\underline{9}$	9	6	3	3	3	1	(0).

The most obvious pattern in the first partition is the triple of 4's. This becomes a triple of 3's one step to the right in the new partition. The new pattern is moved to the right because the new part $\underline{9}$ is inserted in between the 10 and the second 9. The parts larger than 10 in the old partition decrease by one and stay where they are. The parts smaller than or equal to 10 decrease by one and move to the right. The parts less than or equal to 10 in the old partition I call stable parts. In general, a part in a partition is **stable** if it is less than or equal to the size of the partition plus one. Since the new partition in our example has size 9, the parts $10, 9, 9, 6, 3, 3, 3, 1$ are stable. In the next move, these parts move to the right:

#2:	11	10	9	9	6	3	3	3	1
#3:	10	$\underline{9}$	9	8	8	5	2	2	2 (0).

If each number in a set of numbers is decreased by 1 then the differences between any pair of numbers will remain the same. Thus, one way to see patterns more clearly is to *write down the differences between consecutive parts*. If we imagine that there is a 0 after the last part, the differences in partition #1 of our example are as follows:

Differences:	1	1	3	3	0	0	2	1	1	
	∧	∧	∧	∧	∧	∧	∧	∧	∧	·
Partition #1:	12	11	10	7	4	4	4	2	1	(0)

The difference sequences for the next 10 partitions created by playing Bulgarian solitaire are shown in CHART A. In these difference sequences it is evident that numbers and configurations of numbers are moving to the right. For example the pair 3 3 of differences in #1 moves to the right one step at a time until it "hits" the right side in #5. In general the rule of movement is:

Rule 1. The stable differences (differences between stable parts), move one step to the right and remain unchanged except for the last difference which decreases by 1.

Exceptions. There are no exceptions to this rule if it is understood that trailing 0's are to be ignored as in CHART A, #5.

In partition #1 the stable differences are 3, 3, 0, 0, 2, 1, 1 since these are the differences between the stable parts 10, 7, 4, 4, 4, 2, 1, (0). If the last difference is decreased by 1, you get 3, 3, 0, 0, 2, 1, (0), which occurs one step to the right in difference sequence #2.

Large numbers in the difference sequence represent **gaps** in the partition. Thus the 3's in difference sequence #1 of CHART A correspond to the gaps of 3 between the pairs 10, 7 and 7, 4 in partition #1. These **stable gaps** are persistent patterns in the sense that they usually survive for several turns. The differences of 3 in CHART A survive for many turns until they hit the right side and disappear.

This leads us to the following question. What happens when a large gap hits the right side? The answer is that it usually produces a sequence of consecutive 2's on the left side of the stable difference sequence. The gap of 3 in CHART A, #8 produces two consecutive 2's on the left side in CHART A, #11 after three moves. A gap of 4 usually produces three consecutive 2's after four

Partitions

#1	12	11	10	7	4	4	4	2	1			
#2	11	10	9	9	6	3	3	3	1			
#3	10	9	9	8	8	5	2	2	2			
#4	9	9	8	8	7	7	4	1	1	1		
#5	10	8	8	7	7	6	6	3				
#6	9	8	7	7	6	6	5	5	2			
#7	9	8	7	6	6	5	5	4	4	1		
#8	10	8	7	6	5	5	4	4	3	3		
#9	10	9	7	6	5	4	4	3	3	2	2	
#10	11	9	8	6	5	4	3	3	2	2	1	1
#11	12	10	8	7	5	4	3	2	2	1	1	

Differences (gaps)

#1	1	1	3	3	0	0	2	1	1			
#2	1	1	0	3	3	0	0	2	1			
#3	1	0	1	0	3	3	0	0	2			
#4	0	1	0	1	0	3	3	0	0	1		
#5	2	0	1	0	1	0	3	3				
#6	1	1	0	1	0	1	0	3	2			
#7	1	1	1	0	1	0	1	0	3	1		
#8	2	1	1	1	0	1	0	1	0	3		
#9	1	2	1	1	1	0	1	0	1	0	2	
#10	2	1	2	1	1	1	0	1	0	1	0	1
#11	2	2	1	2	1	1	1	0	1	0	1	

CHART A. **The stable numbers are to the right of the black dividers. The new parts in the partitions and the corresponding differences are underlined.**

		Partitions										Differences (gaps)									
#1	24	7	6	6	5	5	4				17	1	0	1	0	1	4			
#2	23	7	6	5	5	4	4	3			16	1	1	0	1	0	1	3		
#3	22	8	6	5	4	4	3	3	2		14	2	1	1	0	1	0	1	2	
#4	21	9	7	5	4	3	3	2	2	1	12	2	2	1	1	0	1	0	1	1
#5	20	10	8	6	4	3	2	2	1	1	10	2	2	2	1	1	0	1	0	1

CHART B. **The stable numbers are to the right of the black dividers. The new parts and new differences are underlined. Rule 2 is illustrated.**

moves and, in general, we have:

Rule 2. A stable gap of g on the extreme right will usually produce $g-1$ consecutive 2's on the left after g moves.

Exceptions. This rule may fail if the number of unstable parts changes.

Thus the last 3 in CHART A, #5 produced only one 2 on the left after three moves because the number of unstable parts changed from 1 to 0.

Let's look at another example where rule 2 holds. Take the partition 24, 7, 6, 6, 5, 5, 4 of $N = 57$ (see CHART B). In this example the first part of each partition is unstable and the other parts are all stable. The stable difference of 4 in difference sequence #1 produces three 2's after four moves as predicted by rule 2.

Rule 2 says that large gaps are usually broken down into sequences of consecutive 2's. The logical question is: What happens to a pair of consecutive 2's when they hit the right side?

Rule 3. A pair of consecutive stable gaps of 2 on the extreme right will usually produce the sequence 2 1 2 on the left after four moves.

Exceptions. This rule may fail when the number of unstable parts changes.

CHART C illustrates rule 3 with the partition 20, 5, 5, 4, 2. What happens to the difference pattern 2 1 2 when it comes to the right edge?

Rule 4. The pattern 2 1 2 on the right of the sequence of gaps usually becomes 2 1 1 2 on the left and this eventually becomes 2 1 1 1 2 and so on.

Exceptions. This rule may fail when the number of unstable parts changes.

What started as a single gap eventually produces difference patterns of the form 2 1 1 ... 1 1 2 in most cases according to rules 2, 3 and 4.

Rule 4 prompted me to think of difference patterns like 2 1 1 1 2 as animals that crawl to the right and grow. These animals should get longer and longer as the game progresses.

		Partitions						Differences (gaps)						
#1	20	5	5	4	2			15	0	1	2	2		
#2	19	5	4	4	3	1		14	1	0	1	2	1	
#3	18	6	4	3	3	2		12	2	1	0	1	2	
#4	17	6	5	3	2	2	1	10	1	2	1	0	1	1
#5	16	7	5	4	2	1	1	8	2	1	2	1	0	1

CHART C. **Rule 3 is illustrated.**

DEFINITION. A **(visible) internal gap** is defined to be one of the following:
a) A single stable difference of 3 or more. This is called a **single gap**.
b) A pair of consecutive stable differences of 2 or more. This is a **double gap**.
c) A pair of stable differences of 2 or more connected by a sequence of one or more 1's. This is a **multiple gap**.

DEFINITION. The **width** of an internal gap is defined to be the number of differences involved. Thus single gaps have width 1, double gaps have width 2, triple gaps have width 3, etc..

For example, suppose you have the stable difference sequence 1 4 0 5 1 1 3 2. This has five visible internal gaps:
a) The differences 4, 5 and 3 are gaps of width 1 (single gaps).
b) The pair 3 2 is a gap of width 2 (a double gap).
c) The sequence 5 1 1 3 forms a gap of width 4 (a quadruple gap).
As an exercise the reader can find the 6 gaps in the stable difference sequence 5 1 2 3 4.

In CHART B the stable difference of 4 becomes three 2's in four moves. In the three intermediate stages however only two 2's are visible. (The 3 in B#2 is equivalent to two 2's.) This can be explained by the fact that there is a "hidden" difference of 2 in each case.

DEFINITION. The **hidden difference** of a partition is defined by the formula:

$$\text{hidden difference} = \text{size of partition} - \text{first stable part} + 1$$

or

$$d = n - s_1 + 1.$$

If there are no stable parts the hidden difference is defined to be the size of the partition plus one ($d = n + 1$).

In CHART B the hidden differences are:

#1	$d = 7 - 7 + 1 = 1$
#2	$d = 8 - 7 + 1 = 2$
#3	$d = 9 - 8 + 1 = 2$
#4	$d = 10 - 9 + 1 = 2$
#5	$d = 10 - 10 + 1 = 1$

The hidden difference can also be computed using the formulas in the following proposition.

PROPOSITION 1. (a) *The sum of the hidden difference and the stable differences is always equal to the size of the partition plus one.*
(b) *If there is at least one stable part then the hidden difference is equal to the new difference in the next partition. (The "new" difference is the one corresponding to the new part in the partition.)*

Proof. Formula (a) follows from the definitions of the various differences. The sum of all the differences is:

$$d + d_1 + d_2 + \cdots + d_m = (n - s_1 + 1) + (s_1 - s_2) + (s_2 - s_3) + \cdots + (s_{m-1} - s_m) + s_m$$
$$= n + 1.$$

where $s_1, s_2, \dots, s_m$ are the stable parts. Formula (b) is obvious because the new part in the next partition is always equal to n, the size of the original partition, and the part after the new part will be $s_1 - 1$ so the new difference will be $n - (s_1 - 1) = d$.

	Hidden Difference	Stable Differences (gaps)
#1	1	1 0 1 0 1 4
#2	2	1 1 0 1 0 1 3
#3	2	2 1 1 0 1 0 1 2
#4	2	2 2 1 1 0 1 0 1 1
#5	1	2 2 2 1 1 0 1 0 1

CHART D. **Difference cycles for the partitions in CHART B.**

Partition			Differences			Difference cycle			
6			6				2		
5	1		4	1			2	1	
4	2		2	2			1	2	
3	2	1	1	1	1	1	1	1	1

CHART E. **The sequence of partitions and corresponding differences and difference cycles beginning with the partition 6 of $T_3 = 6$.**

According to Proposition 1(b) the hidden differences in CHART A are 0, 0, 0, 2, 1, 1, 2, 1, 2, 2. The hidden difference for the last partition is $d = 11 - 12 + 1 = 0$.

When you add the hidden difference to the sequence of stable differences the patterns make much more sense and you can see clearly how the patterns change with each move. For example take CHART B and write the hidden differences in front of the stable differences (see CHART D). I call each of these five sequences a **difference cycle**. The missing 2's are visible and there is a clear rule relating each cycle to the next.

Rule 5. (1) If the last difference in the difference cycle is 2 or more, then usually it decreases by 1 and a 2 appears in the first position (e.g., $xyz4$ becomes $2xyz3$).
(2) If the last difference in the difference cycle is 1, then it usually just moves to the first position (e.g., $xyz1$ becomes $1xyz$).

Exceptions. The rule may fail when (a) the number of unstable parts changes, or (b) when the last two differences in the difference cycle are 0, 1.

This rule follows from rule 1 and Proposition 1(a). Rules 2, 3 and 4 are consequences of rule 5.

An interesting example of rule 5 occurs when you take a partition of size 1 (a single number). Take, for example, the partition 6 of $N = 6$ (see CHART E). In the last step, rule 5 is broken because the number of unstable parts changes from 1 to 0. The difference cycles for CHART A are given in CHART F. In the last step rule 5 is broken because #10 ends in 0 1.

With the addition of the hidden difference new gaps are revealed. For example, in F#9 the sequence 2 1 2 is a hidden internal gap since the first difference 2 is "hidden." This sequence becomes visible in one move in F#10. The sequence 2 1 2 of F#9 comes from the sequence 3 : 1 2 which occurs in F#8 when it is written twice in a row (12111010103 : 12111010103). I call this an external gap. In general, an **external gap** is defined to be a sequence of consecutive differences in the difference cycle written twice so that the first and last differences in the sequence are at least 2 and the others are exactly 1. Thus F#9 also contains the external double gap 2 : 2.

Another example is the difference cycle 2 1 1 2 0 3 1. This has two internal gaps: the single gap 3 and the hidden quadruple gap 2 1 1 2. The first difference and the last two differences form an external triple gap 3 1 : 2.

#1			0	3	3	0	0	2	1	1			
#2		0	1	0	3	3	0	0	2	1			
#3	0	1	0	1	0	3	3	0	0	2			
#4	2	0	1	0	1	0	3	3	0	0	1		
#5		1	0	1	0	1	0	3	3				
#6	1	1	1	0	1	0	1	0	3	2			
#7	2	1	1	1	0	1	0	1	0	3	1		
#8	1	2	1	1	1	0	1	0	1	0	3		
#9	2	1	2	1	1	1	0	1	0	1	0	2	
#10	2	2	1	2	1	1	1	0	1	0	1	0	1
#11	0	2	2	1	2	1	1	1	0	1	0	1	

CHART F. **Difference cycles for the partitions in CHART A.**

Let's take another example. Suppose you have the difference cycle 1 2 1 1. This has an external gap of width 4 which you can see if you write the cycle twice: 1 2 1 1 : 1 2 1 1. The external gap is 2 1 1 : 1 2. This is a quadruple gap (width 4) since the first and last numbers are actually the same difference. (I like to think of this as a gap "eating its own tail.")

One last example. Take the partition 18 10 of 28. Since there are no stable parts, the difference cycle consists of the single hidden difference 3. This has two gaps: the single (hidden) internal gap 3 and the single external gap 3 : 3.

For each difference cycle, there are only three kinds of gaps: visible internal gaps, hidden internal gaps, and external gaps. Every gap has a width (w), a position (p), and a height (h). The width of a gap has already been defined; I will now explain what its position and height are.

Every part in a partition has a position. The first part has position 1, the second part has position 2, and so on. The differences also have positions. The difference between the ith and the $(i+1)$st parts has position i. In the difference cycle, the hidden difference comes before the first stable difference so its position is always equal to the number of unstable parts. For example, take the partition $4,4,3,3,1$. The stable differences are $0,1,0,2,1$ with positions $1,2,3,4,5$, respectively, so the hidden difference of 2 has position 0. For the partition $12,9,4,3,1$ the stable differences $1,2,1$ have positions $3,4,5$, respectively, so the hidden difference of 2 has position 2.

DEFINITION. The **position** of a gap is defined to be the position of the "last" difference in the gap.

For example, take the partition $16,9,7,6,3,3,1$. This has difference cycle 1 1 3 0 2 1 with positions $2,3,4,5,6,7$, respectively. There are two gaps: 3 and 2 1 : 1 1 3. Each has position 4 since the "last" difference in each is 3 which has position 4.

If a part x in a partition has position p then its **height** is defined to be $x+p$. For example in the partition $16,9,7,6,3,3,1$ the heights of the parts are $17,11,10,10,8,9,8,(8)$. The corresponding differences are defined to have the same heights. Thus in the difference cycle 1 1 3 0 2 1 (of the pattern just considered) the differences have heights $11,10,10,8,9,8$, respectively (see CHART G).

parts:	16	9	7	6	3	3	1	(0)
+ positions:	1	2	3	4	5	6	7	8
= heights:	17	11	10	10	8	9	8	8
difference cycle:		1	1	3	0	2	1	

CHART G

DEFINITION. (1) The **height** of a single internal gap is defined to be the height of the next part (e.g., the single gap 3 in CHART G has height 8).
(2) The height of a multiple gap or single external gap is defined by the formula:

$$\text{height} = \text{“last” difference in gap} + \text{height of next part} - 2$$

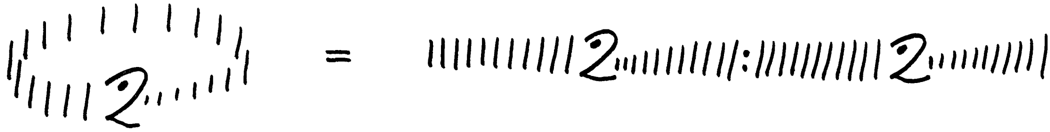
(e.g., the gap 2 1 : 1 1 3 in CHART G has height $h = 3 + 8 - 2 = 9$).

Let's take another example. The partition $18,10$ has difference cycle 3 so it has two gaps: 3 and 3 : 3. These have heights 3 and 4 as indicated in CHART H.

parts:	18	10	(0)
+ positions:	1	2	3
= heights:	19	12	3

difference cycle: 3
height of 3 = height of third position = 3
height of 3 : 3 = 3 + 3 − 2 = 4

CHART H



Proofs of the Theorems

We can now prove Theorem 3, using the definitions given in the previous section. We must show that every nonrepeating partition of $N \leqslant T_k$ has at least one gap with height $h \leqslant k$.

Proof of Theorem 3. Suppose the given partition is $A = (a_1, a_2, \dots, a_n)$, the a_i written in decreasing order as usual. The heights of the n parts are $a_1 + 1$, $a_2 + 2, \dots,$ $a_n + n$ and the next position has height $n + 1$. Let h_0 be the smallest of these $n + 1$ heights and let h_1 be the largest. Then I claim that $h_0 \leqslant k$ and $h_1 \geqslant h_0 + 2$.

To see that $h_0 \leqslant k$, suppose otherwise. Then $a_1 \geqslant k$, $a_2 \geqslant k - 1, \dots,$ $a_n \geqslant k + 1 - n$ and $n \geqslant k$. This can't happen unless $N = T_k$ and A is the winning partition. To prove that $h_1 \geqslant h_0 + 2$, suppose otherwise. Then every height is either h_0 or $h_0 + 1$. This means each $a_i = h_0 - i$ or $h_0 - i + 1$ and $n = h_0$ or $h_0 - 1$. By Theorem 1(b) this can only happen when $h_0 = k$ and A is a repeating partition.

I now claim that there is a gap with height exactly h_0. To find this gap you take the smallest $p_0 \leqslant n + 1$ so that position p_0 has height h_0 and take the largest $p_1 \leqslant n + 1$ so that position p_1 has height $h_0 + 2$ or more. There are two cases: either p_1 is a stable position or it isn't (p_0 is always stable).

If p_1 is unstable then there must be a gap with position $p_0 - 1$ and height h_0 and this gap must be hidden. If p_1 is stable then there are again two cases: either there exists a third position p_2 with height h_0 so that $p_1 < p_2 \leqslant n + 1$ or else there doesn't. If p_2 exists then, taking p_2 to be minimal, there must be a visible internal gap with position $p_2 - 1$ and height h_0. If p_2 doesn't exist then there must be a hidden gap with position $p_0 - 1$ and height h_0.

Charts C and B illustrate the occurrence of all three cases considered in the proof to show there is a gap with height h_0. Partitions 2,3,4 of Chart C are examples of the first case, while partitions 1 and 5 of Chart C are examples of the second case. Partition #3 of Chart B is an example of the last case.

It remains to prove Theorem 2, which follows from the following two lemmas.

Lemma 1. *If a gap in a partition has width* w, *position* p, *and height* h, *then* $1 \leqslant w \leqslant h - 1$ *and* $0 \leqslant p \leqslant h - 1$.

This lemma implies that there are exactly $h(h-1)$ possible values for the ordered pair (w, p). This ultimately leads to the term $h(h-1)$ in Theorem 2 and the term $k(k-1)$ in the Bulgarian solitaire conjecture.

Proof. From the definitions of w and p it is clear that $w \geqslant 1$ and $p \geqslant 0$. From the definition of h it is clear that h is at least equal to the height of position $p + 1$ $(h \geqslant p + 1 + x)$. This makes $p \leqslant h - 1$. To show that $w \leqslant h - 1$ you have to look at the three cases of a single gap, a multiple internal gap and a multiple external gap.

Case 1 (single gap). A single gap has $w = 1$ but the height of any gap is always at least 2.

Case 2 (multiple internal gap). Suppose the gap is given by the difference sequence d_1, $d_2, \dots, d_w$. These represent differences between $w + 1$ parts $a_1, a_2, \dots, a_{w+1}$. By definition $h = d_w$ + (height of $a_{w+1}) - 2 \geqslant$ height of $a_{w+1} \geqslant a_{w+1} + w + 1 \geqslant w + 1$.

Case 3 (multiple external gap). The height of any external gap is always equal to the size of the partition plus the number of unstable parts $(h = n + u)$. But w is always less than $n + u$.

DEFINITION. The **weight** of a gap is defined to be the ordered pair (w, p) where w is the width and p is the position of the gap. Weights are ordered lexicographically, so, for example, $(3,2) > (2,5)$ but $(3,2) < (3,4)$.

LEMMA 2. *Suppose A is a partition of N which, after one move of Bulgarian solitaire, produces the partition B. Then*

(1) *Every gap in partition B comes from a gap (its "parent") in partition A.*
(2) *The height of every gap in B is greater than or equal to the height of its parent ($h' \geqslant h$).*
(3) *The weight of every gap in B is strictly greater than the weight of its parent ($(w', p') > (w, p)$).*

I should first explain the terminology in this lemma. Take, for example, A and B to be the partitions with difference cycles given by F#10 and F#11. I would like to think of the gaps 2 2 that occur in F#10 and F#11 as being the "same" gap which has "moved." However, it is more rigorous to say that gap 2 2 in F#10 produces another gap of 2 2 in F#11 and call the first gap the "parent" of the second gap.

Before I prove Lemma 2, I will explain why Lemmas 1 and 2 imply Theorem 2 and thus the Bulgarian solitaire conjecture.

Take all the gaps in partition A which have height h or less. Of these gaps choose the one with the smallest weight, say, (w, p). Do the same thing with partition B and you get a second gap with weight (w', p'). I claim that $(w', p') > (w, p)$. This is because the "parent" of the second gap has weight, say, (w'', p''), which is less than (w', p') by Lemma 2(3). The height of this parent gap is h or less by Lemma 2(2), and so $(w, p) \leqslant (w'', p'') < (w', p')$.

Since the smallest weight of a gap with height h or less is always increasing and there are exactly $h(h-1)$ possibilities for such a weight by Lemma 1, you exhaust all the possibilities after $h(h-1)-1$ moves, and the next move must eliminate all gaps with height h or less.

The only thing left now is to prove Lemma 2. I will start by proving this lemma in the case when rule 5 governs the movement of the difference cycle. Later I will explain what happens for the exceptions to rule 5.

Suppose you have a difference cycle $x\ y\ 2\ 1\ 1\ 2\ z$. For the next z moves the last difference will decrease one at a time until it reaches 0. During that time, the gap 2 1 1 2 moves one step to the right each turn (its position increases by 1 each turn) but it stays the same width and at the same height. At the end of z moves you have $* \ * \cdots x\ y\ 2\ 1\ 1\ 2$. After one more move, according to rule 5, you get $2\ * \ * \cdots x\ y\ 2\ 1\ 1\ 1$. The gap 2 1 1 2 changes to 2 1 1 1 : 2. The width increases by 1. In general the rule is:

Rule 6. If a multiple gap or external single gap has width w, position p, and height h, then its position will increase one step at a time with w and h fixed until p reaches $h-1$. When $p = h-1$, the next move will increase w by 1 ($w' = w+1$), change p to $u =$ the number of unstable parts in the partition ($p' = u$), and increase h by u ($h' = h + u$).

Exceptions. This rule holds only when rule 5 applies.

This rule explains what happens to all gaps except single internal gaps. To see what happens to these gaps let's go back to CHART B. In B#1 the single gap 4 has position 7 and height 8. In B#2 the difference cycle is 2 1 1 0 1 0 1 3 so there are two gaps: the single gap 3 has $p = 8$, $h = 9$. The double gap 3 : 2 has $p = 1$, $h = 9$. In B#3 the difference cycle is 2 2 1 1 0 1 0 1 2. There are again two gaps, the double gap 2 : 2 with $p = 1$, $h = 10$ and the double gap 2 2 with $p = 2$, $h = 9$. The single gap 4 produces two gaps but the single gap 3 produces only one gap. The double gap 3 : 2 changes according to rule 6.

Rule 7. Suppose you have a single gap with position p and height h. With each move the position p increases by 1 keeping $w = 1$ and h fixed until p reaches $h-1$. When $p = h-1$ there are two cases:

(1) If $g = 3$ then the single gap 3 produces a double external gap 2 : 2 with position $p' = u$ and $h' = h + u$. (Of course $w' = 2 = w + 1$.)

(2) If $g \geqslant 4$ then the single gap g produces two gaps: a single gap $g-1$ with $p' = p+1$, $h' = h+1$ and a double external gap $g-1:2$ with $p' = u$ and $h' = h+u$.

Exceptions. This rule holds only when rule 5 holds.

The conclusion is that in each move where rule 5 holds each gap in a partition is replaced by one or two gaps whose heights are larger than or equal to the height of the original gap. Furthermore, this accounts for all gaps in the new partition. In other words, Lemma 2 holds whenever rule 5 applies. I should note that rules 6 and 7 follow from rule 5.

There are two exceptions to rule 5. One of these occurs when the difference cycle ends with 0 1. This occurs, for example, in CHART F#4 and #10. Both of these are examples of the following.

Rule 8. Suppose that the difference cycle of a partition ends with x number of 0's and a 1. To find the difference cycle for the next partition you move the cycle one step to the right, bring the 1 to the empty space on the left, and then diminish the differences on the left by x. The last number so diminished is the new hidden difference.

Exceptions. None, assuming that $x \geqslant 1$.

For example, take the difference cycle 2 3 0 0 0 1. At first this becomes 1 2 3 0 0 0. The first 0 eliminates the 1 and the other two 0's eliminate the 2. Since the 2 was the last number diminished, the new difference cycle is 0 3. Another example is the difference cycle 4 3 0 0 1. This becomes 3 3 because you take away 2 from the left end of 1 4 3.

What happens to the gaps in a partition when rule 8 applies?

Rule 9. If the last two differences in the difference cycle of a partition are 0 1, then the gaps in the partition fall into three categories.

(1) "most" gaps simply move one step to the right, staying at the same width and height ($w' = w$, $h' = h$, $p' = p+1$).

(2) Gaps too far to the left simply disappear.

(3) There may be one gap on the "border" which is not quite far enough to the left to disappear. If this gap is a multiple gap it moves one step to the right and becomes larger ($w' > w$, $h' = h$, $p' = p+1$). If the gap on the border is a single gap, then it produces one or two gaps: (a) a single internal gap with $h' = h$, $p' = p+1$, and (b) a multiple gap or single external gap with $h' = h$, $p' = p+1$.

Exceptions. None.

For example, suppose you have the difference cycle 2 8 1 2 0 0 0 0 1. This has three gaps: the single gap 8, the double gap 2 8, and the triple gap 8 1 2. According to rule 8, the next difference cycle is 7 1 2. The relative positions are:

```
2 8 1 2 0 0 0 0 1
    7 1 2.
```

The gap 8 1 2 becomes 7 1 2 one step to the right. The gap 2 8 is gone. The single gap 8 produces two gaps: 7 and 2:7. These each have position 7 and height 11 whereas the original single gap 8 had position 6 and height 11. (How do I know this?)

Another example is the partition 1 1 1 1 1 1 of $N = 6$. This has difference cycle 6 0 0 0 0 0 1. The next difference cycle is the single number 2. The single gap 6 falls into category (3) of rule 9 since it produces (b) a single external gap with $h' = h$, $p' = p+1$.

Whenever the difference cycle ends in 0 1, rule 8 applies and thus the gaps in the partition change according to rule 9. In particular Lemma 2 holds for each gap. I should note that rule 9 follows from rule 8 by a careful case by case analysis and rule 8 follows from the fact that the next partition has size x less than the original partition ($n' = n - x$).

The other exception to rule 5 occurs when the number of unstable parts changes ($u' \neq u$). The number gets larger only when the last three differences are 0 0 1 (see, for example, CHART A).

This was dealt with in rules 8 and 9. The only remaining case is when the number of unstable parts decreases ($u' < u$).

Rule 10. Suppose that the number of unstable parts decreases by x ($u' = u - x$) and the last difference in the difference cycle is 1 (e.g., $a\ b\ c \dots z\ 1$). Then the new difference cycle is $1\ a\ b\ c \dots z$ preceeded by x zeros.

Exceptions. None, assuming that $x > 0$.

This rule follows from Proposition 1(a) and the fact that the size of the partition remains the same ($n' = n$). CHART F has two examples of this in #1 and #2.

Rule 11. When rule 10 holds, all internal gaps simply move one step to the right, keeping the same width and height and all external gaps disappear (e.g., 1 3 1 2 1 becomes 0...0 1 1 3 1 2, killing the external gap 2 1:1 3).

Exceptions. None, assuming that $x > 0$.

This rule follows easily from rule 10 and it clearly implies that Lemma 2 holds whenever rule 10 applies.

Rule 12. Suppose that the number of unstable parts decreases by x ($u' = u - x$) and the last difference in the difference cycle is 2 or more. To find the new difference cycle, you move the differences in the old cycle one step to the right, decrease the last difference by 1, and insert on the left side $x + 1$ nonnegative numbers which add up to 2.

Exceptions. None.

This rule follows from Proposition 1(a) and the fact that the size of the partition increases by 1 ($n' = n + 1$). CHART E has an example of this rule. Another example is the partition 9 8 8 6 3 2. This has $u = 3$ and difference cycle 1 3 1 2. The next partition is 8 7 7 6 5 2 1 with $u' = 0$ and difference cycle 0 1 0 1 1 3 1 1. The four new differences 0 1 0 1 add up to 2. The original partition had three gaps. The new partition has only one. The single gap 3 is the only one that survives.

The same difference cycle 1 3 1 2 could change in several ways. For example, 2 0 1 3 1 1, 0 2 1 3 1 1 and 1 1 1 3 1 1 are all possible. Each one of these has two gaps. In the first case the external gap 2:1 3 of the original cycle dies and the gap 3 1 2 grows into 3 1 1:2 in the usual way (as in rule 6). In the second case the gap 3 1 2 dies and the other two gaps simply move one step to the right. In the last case, the gaps 3 1 2 and 2:1 3 produce the gap 3 1 1:1 1 1 3. I would like to interpret this by saying that the gap 3 1 2 dies and the gap 2:1 3 grows and moves one step to the right, making this example the same phenomenon as in rule 9(3).

If you carefully examine all possibilities for rule 12 you will see that each gap changes according to rules already discussed. Consequently the movement of gaps is always governed by rules 6, 7, 9 and 11, and Lemma 2 holds in all cases.

This concludes my proof of the Bulgarian solitaire conjecture as stated in Corollary 1. However, there are a few other interesting results about Bulgarian solitaire that follow from this approach.

Suppose that $T_k \geqslant N > T_{k-1}$. Then given any nonrepeating partition of N you can examine the gaps in the partition and tell approximately how long it will take to reach a repeating partition. You know by Theorem 3 that the partition has gaps of height $\leqslant k$. Take the smallest weight that occurs for these gaps and call it the **weight** of the partition. Then Lemma 2 implies:

THEOREM 4. *The weight of a partition increases in lexicographic order with each move of Bulgarian solitaire.*

For example, take CHART C. The weights of the five partitions are: (2,5), (3,1), (3,2), (3,3), (3,4). There are 35 weights larger than (3,4), so the game must end in 36 moves or less. In fact, the game ends in 34 moves because the weights (3,7) and (4,0) are not encountered.

COROLLARY 2. *If $T_k \geqslant N > T_{k-1}$ and a partition of N has weight (w, p), then there are exactly $(k-w-1)k+(k-p-1)$ weights larger than (w, p) so a repeating partition is reached in $(k-w-1)k+(k-p-1)+1 = k^2 - wk - p$ moves or less.*

If $N \neq T_k$, you can do slightly better because the weight $(k-1, k-1)$ never occurs. This leads to the following question: *For each value of N can you determine exactly which weights are possible for the partitions of N?* This is related to the question: *What is the maximum number of moves required to make each partition of N into a repeating partition?* (This question is still unanswered when N is not triangular. I don't even know a conjectured formula.)

To close, I give an algorithm to compute the weight of any partition.

(1) Input: n = the size of the partition.
$a_1, a_2, \ldots, a_n$ = the parts of the partition.
(2) Let $N = a_1 + a_2 + \cdots + a_n$.
(3) Let k be minimal so that $N \leqslant k(k+1)/2$.
(4) Let $a_{n+1} = 0$.
(5) Compute u = the number of unstable parts: If $a_1 \leqslant n+1$, then let $u = 0$. Otherwise let u be maximal so that $a_u > n+1$.
(6) For $i = u+1, u+2, \ldots, n$ let $d_i = a_i - a_{i+1}$.
(7) Let $d_u = n+1-a_{u+1}$.
(8) Look for simple internal gaps: For $u \leqslant p \leqslant n$ check if $d_p \geqslant 3$. If $d_p \geqslant 3$, then check if $a_{p+1}+p+1 \leqslant k$. If p is minimal so that $d_p \geqslant 3$ and $a_{p+1}+p+1 \leqslant k$, then the weight of the partition is $(1, p)$.
(9) If $n+u > k$ then go to step 12.
(10) Check if $u = n$. If this is so then the partition has weight $(1, u)$.
(11) See if the partition has an external gap: Let p be minimal so that $d_p \neq 1$ and $u \leqslant p \leqslant n$. Let q be maximal so that $d_q \neq 1$ and $u \leqslant q \leqslant n$. If d_p and d_q are both nonzero, then the partition has an external gap with position p, height $h = n+u$, and width $w = n-u+2-\max(q-p, 1)$. Otherwise the partition has no external gaps.
(12) Find the multiple internal gaps: For $u < p \leqslant n$ check if $d_p > 1$. For each $d_p > 1$, let q be maximal so that $d_q \neq 1$ and $u \leqslant q < p$. If $d_q > 1$, then you have an internal gap with position p, height $h = a_p + p + 1$, and width $w = p - q + 1$. If $d_q = 0$ or if q doesn't exist, there is no internal gap with position p.
(13) The weight of the partition is the smallest weight (w, p) of a gap found in steps 11 and 12 with height $h \leqslant k$.

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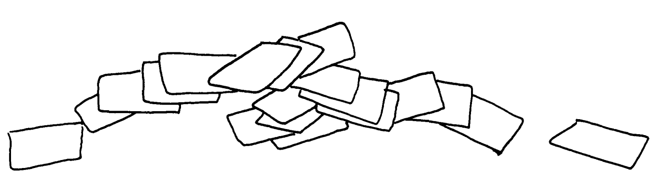

Constructive Mathematics

Recent advances in constructive mathematics draw attention to the importance of proofs with numerical meaning.

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An age-old controversy in mathematics concerns the necessity and the possibility of constructive proofs. While moribund for almost fifty years, the controversy has been rekindled by recent advances which demonstrate the feasibility of a fully constructive mathematics. This nontechnical article discusses the motivating ideas behind this approach to mathematics and the implications of constructive mathematics for the history of mathematics.

For over a hundred years the controversy over constructivity has been simmering silently beneath the surface of mathematics, occasionally erupting into full-blown battle, but never reaching a settlement. Still the conflict continues. Today, while the vast majority in mathematics use nonconstructive methods, a small minority persist in the struggle to bring constructive methods into general use.

During a previous episode in this controversy, Einstein asked, "What is this frog-and-mouse battle among the mathematicians?" [**20**, p. 187]. One might ask the same today. To begin with a brief answer which is elaborated below, constructive proofs are those which ultimately reduce to finite constructions with the integers $1, 2, 3, \ldots$. Such proofs are said to have *numerical meaning.* In *constructive* mathematics, numerical meaning is central. In contrast, *classical* mathematics, which is dominant today, admits methods which are not in essence finite, with the result that classical proofs often lack numerical meaning.

Recent advances in constructivity, which point to a final resolution of the problem, stem from Errett Bishop's 1967 book, *Foundations of Constructive Analysis* [**1**]. A growing number of mathematicians now work in accord with the general principles proposed and developed by Bishop.

The present article discusses the basic ideas which motivate this group of modern constructivists. The treatment deals with no technical details, but rather with fundamental human attitudes towards mathematics, its meaning, and its purpose. (For a thorough technical discussion, see [**25**].) The author is indebted to Y. K. Chan, Michael Goldhaber, Keith Phillips, and the late Errett Bishop for critical readings of early drafts of this article.

Historical background

The present controversy in mathematics has traces even in the early history of mathematics.

> *Not geometry, but arithmetic alone will provide satisfactory proof.*
> Archytas, ca. 375 B.C. [**13**, p. 49]

Archytas refers to an ancient controversy concerning geometric and arithmetic proofs. Certain aspects of this controversy correspond roughly to the present controversy between classical and constructive proofs.

Throughout the history of mathematics, both constructive and nonconstructive tendencies are found. The Pythagoreans tried to reduce all mathematics to numbers. Plato introduced, through his theory of forms, an idealistic approach to philosophical problems which pervades all classical mathematics; he taught that truth exists independently of humans, who must seek it through dialectic. Aristotle began the systemization of logic; his *principle of excluded middle* (discussed below) is a major cause of nonconstructivities when applied to the mathematical infinite. Gauss first gave his complex numbers a geometric representation, but later considered this inadequate and gave an arithmetic formulation. In the late nineteenth century a violent attack on nonconstructive methods was led by Leopold Kronecker in Berlin; his conviction was: "God made the integers, all else is the work of man" [**13**, p. 988]. At the turn of the century, Henri Poincaré in Paris strongly advocated constructive methods; interested in applications, he criticized the classical approach, saying, "True mathematics is that which serves some useful purpose" [**19**].

Although classical mathematics has long been dominant, it was severely challenged during the early part of the century. The challengers, led by L. E. J. Brouwer in Amsterdam, were critical of current practice and called for a new beginning, a careful reconstruction of the basic mathematical framework. The defense, content with classical methods, was led by David Hilbert in Göttingen.

Brouwer (1881–1966) demonstrated that classical mathematics is deficient in numerical meaning. Beginning in 1907, he devoted much of his life to attacking classical methods whose validity he questioned, showing that these methods did not produce mathematical objects which are explicitly constructed and which ultimately reduce to the integers. Brouwer held that in the "constructive process... lies the only possible foundation for mathematics" [**13**, p. 1200]; [**5**].

Hilbert (1862–1943) was a leader in the development of classical mathematics. His solution of "Gordan's Problem" in 1888 had accelerated the growth of modern mathematics. His proof, however, was nonconstructive. Paul Gordan himself, who had for twenty years tried to solve the problem, exclaimed, "Das ist nicht Mathematik. Das ist Theologie!" [**20**, p. 34]. Leading the classical defense against the constructivists, Hilbert angrily complained that "forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists" [**13**, p. 1204].

Hermann Weyl in Zürich strongly supported Brouwer and the constructivists. He charged nonconstructive proofs with lack of significance and value, saying that classical analysis is "built on sand" [**13**, p. 1203].

The Brouwer-Hilbert debate, and the subsequent work of the Brouwerian school, was devoted more to logical, philosophical, and foundational considerations, than to positive constructive developments. A significant advance in the latter direction was made in 1967 by Errett Bishop (1928–1983) in California; his book [**1**] succeeds in developing a large portion of analysis in a realistic, constructive manner.

The central idea in modern constructive mathematics is

BISHOP'S THESIS: *All mathematics should have numerical meaning* [**1**, p. ix].

The significance of this depends upon two essential conditions. First, classical mathematics is deficient in numerical meaning. Second, it is in fact possible to give most mathematics numerical meaning. The first was demonstrated by Brouwer. Although he and his followers also made a great effort to demonstrate the second, and did constructivize certain isolated portions of mathematics, they unfortunately introduced unnecessary idealistic elements into much of their work. Thus Brouwer's main contribution, of crucial significance to the development of mathematics, was his *critique of classical mathematics*. The second step, the systematic *constructive development of mathematics*, was begun in 1967 by Bishop. A small number of workers now continue this development, in constructive algebra [**12**], [**21**], [**24**], [**26**]; constructive analysis [**2**], [**3**], [**4**], [**6**], [**9**], [**10**], [**15**], [**16**], [**17**]; constructive probability theory [7], [**8**], and constructive topology [**22**].

Bishop's maxim, "When a man proves a number to exist, he should show how to find it," expresses the constructivist thesis. His book, written to demonstrate the feasibility of "a straightforward realistic approach to mathematics," rebuilds the basics of analysis in a fully constructive manner and provides the framework and methods for continuing further work "to hasten the inevitable day when constructive mathematics will be the accepted norm" [**1**].

Classical vs. constructive mathematics

To avoid a common misunderstanding, it should be stressed that from the constructivist position, classical mathematics does not appear useless, but merely limited. The limitation is crucial, but not fatal. A theorem proved by classical methods is merely incomplete; the degree of incompleteness varies to both extremes. A classical theorem may exhibit no numerical meaning, and there may be little chance of extracting any numerical meaning from it. At the other extreme is the classical theorem that is constructive as it stands, or becomes constructive after some quite minor reformulations. Situated between these extremes, most classical theorems are neither constructively valid nor completely devoid of numerical meaning. Such a theorem must be significantly modified and rephrased to show its true constructive content, and a considerable amount of work must be done to find a constructive proof. Often several new theorems are obtained, revealing different constructive aspects of the classical theorem which were hidden by its classical presentation.

An example will show how a classical theorem can lead to new constructive theorems. Consider the intermediate value theorem which says that *a continuous curve, which is somewhere below the axis and somewhere above, must somewhere cross the axis*. This theorem is not constructively valid; there is no general finite procedure for constructing a crossing point. Furthermore, a counterexample, of a special type due to Brouwer, convinces us that we will never find such a procedure. What can be done? We certainly do not want to discard such a beautiful theorem!

There are two main methods for salvaging constructive theorems from constructively invalid classical theorems. The first method weakens the conclusion, the second strengthens the hypotheses. In each case we must find a constructive proof. Although the resulting theorem does not sound as strong as the original, it is in a deeper sense much stronger—it has numerical meaning.

Applying these methods to the continuous curve theorem, we first weaken the conclusion. We find that we are able to construct, for any small positive number ε, a point of the curve which has a distance less than ε from the axis.

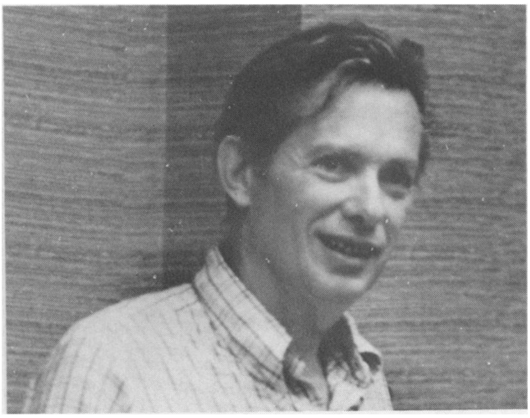
The numerical content of this constructive theorem is clear. What was the content of the original (constructively invalid) classical theorem? The theorem by itself merely makes a statement; it says that there exists a point of the curve which lies on the axis. In what sense, we ask, does such a point exist? Can we actually find the point? Is there a method for constructing it? The mere statement of the classical theorem does nothing to answer these questions. When we examine the proof to see what is actually proved, we find the following: Assuming that no point of the curve lies on the axis, by a certain series of deductions we obtain a contradiction. Thus "existence" in the classical theorem merely means "nonexistence is contradictory." In sharp contrast, our constructive proof of the modified theorem contains the actual construction of a point.

To obtain a second constructive theorem, we strengthen the hypotheses, adding the condition that the curve is defined by a polynomial. This extra hypothesis, while restricting the scope of the theorem, enables us to obtain the original conclusion in full constructive force. We can construct a point of the polynomial curve and prove that it lies exactly on the axis. Various other constructive theorems are also obtained [**1**, p. 59].

For the intermediate value theorem, however, the first constructivization, which finds a point of the curve within ε of the axis, is more important. This is because of its wider applicability, and because, in general, finding solutions to problems "within ε," rather than "exactly," is quite sufficient. This is especially clear when one considers practical applications of mathematics.

What does the classical proof demonstrate? It shows that the existence of such a continuous curve, together with a proof that no point lies on the axis, would lead to a contradiction. This is a useful aspect of the classical proof; we need not waste time trying to construct such a curve. Thus, while the classical theorem may have no constructive affirmative conclusion, it does have practical (although limited) usefulness in directing further constructive effort.

Thus, far from being useless from a constructive viewpoint, classical mathematics serves as an invaluable guide in building the first stages of a fully constructive mathematics. To a large extent



Errett Bishop

this has already been done. Bishop's book contains a complete constructive development of basic analysis: the real numbers, calculus, complex analysis, metric spaces, measure and integration, linear spaces, and more, are constructivized. Others have extended this work and have also begun to constructivize algebra, topology, and probability. Yet there is still much more classical mathematics in need of constructivization.

After the classical results in a given subject have been constructivized as far as possible, then constructive mathematics proceeds to develop the various constructive aspects of the subject which have been uncovered. This enrichment of a mathematical subject is a result of distinguishing between what is constructive and what is not, and is analogous to the enrichment of mathematics which resulted when mathematicians began to distinguish, for example, between infinite series which were convergent, and those which were not.

The principle of excluded middle

The continuous curve problem illustrates the foremost cause of nonconstructivity and lack of numerical meaning in classical mathematics: widespread use of the principle of excluded middle, which says that any meaningful statement is either true or false.

Great care is required with the "either-or" construction, which appears in the excluded middle, and which may be subject to varied interpretations and meanings. Suppose you are going to lunch with a friend and you wish to know whether to take your umbrella. If, using the principle of excluded middle, he or she tells you "*Either* it will rain this noon *or* it will not," you will find this information useless. On the other hand, if *either* you are told "It will rain this noon," *or* you are told "It will not rain this noon," then you will have received valuable constructive information. (This example may give occasion for reflection upon the sort of information, and the degree of certitude, that is found in mathematics, as compared with, for example, meteorology. In fact, mathematical proofs do give predictions. They predict that if certain numbers, with certain relationships between them, are used in performing certain calculations, then the resulting numbers will exhibit certain new relationships.)

Some statements, such as "there exists a prime number between 17,000,000,000 and 17,000,000,017", are in fact constructively either true or false. This is because there is a finite procedure for deciding. For this reason, we might be able to use an indirect proof, a "proof by contradiction." If the assumption that there is no prime number in the specified interval leads to a contradiction, then it would be constructively valid to conclude that there does exist such a prime number. The essential ingredient to such an indirect proof is the prior possession of the finite procedure which either finds the desired prime number or proves there is none. The indirect proof shows that the second alternative is contradictory, and thus *predicts* that the finite procedure, when carried out, will lead to the first alternative, the construction of a prime number in the specified interval. It was for such finite situations only that Aristotle formulated his rules of logic, especially the principle of excluded middle. Nonconstructivities arise when these rules are used indiscriminately in modern mathematics, the science of the infinite.

Now recall our continuous curve. The statement "there exists a point of the curve which lies on the axis" admits of no finite method to determine its truth. There are infinitely many points of the curve. Even for a single point there is no finite method for deciding whether it lies on the axis. In contrast to the above statement about prime numbers, here there is no prior finite procedure for determining one of two alternatives, and thus an indirect proof is constructively invalid.

To the constructivist, use of the principle of excluded middle in infinite situations leads only to pseudoexistence, in the sense that nonexistence is contradictory, rather than existence which stems from a construction. It is the latter type of existence which is appropriate to finite man. While the objects of constructive mathematics are solid objects created by finite constructions, many of the objects of classical mathematics appear as disembodied entities born of questionable logical laws.

Applications of mathematics

A constructive proof, which actually constructs a point with certain properties, has a practical advantage over a classical proof, which merely shows that it is unthinkable that the desired point is nonexistent.

Wandering in the Sahara, would we be content with a nonconstructive proof of the existence of an oasis? Would our parched throats be satisfied with a theorem which asserts that water exists somewhere in the desert, but gives us no clue whatever as to where? Or would we prefer a constructive drink? The classical camel tries to reassure us that there does indeed exist an oasis, but it cannot tell us the direction or the distance. The constructive camel, on the other hand, gives us a direction which, while it may not lead exactly to the oasis, will lead as close as desired. It does give us a point of the compass to follow, and an approximate distance. We might ask it to calculate the direction so as to pass within twenty meters of the oasis. Although it could calculate the direction so as to pass within one millimeter of the exact center of the oasis, this calculation, while still finite, would probably take much longer, wasting precious time. In what sense does the classical camel give the direction? The classical camel asserts that the exact direction exists, but it would take it an infinite amount of time to calculate it, even to find a first approximation. Unless our goatskins happen to contain an infinite amount of water, we might find this classical calculation a bit lengthy.

Applications of mathematics are similar to the Sahara problem. No scientist would be content to learn that a solution to his mathematical problem exists, but that there is no way to calculate it. Thus all experience tends to indicate that any mathematics that is applicable must be constructive. Although there seem to be a few applications of nonconstructive mathematics to theoretical physics, it is likely that it will be the constructive content of these applications which will be useful when the theory reaches the point of experimental verification.

If it is the constructive content of mathematics which is applicable, then since so much mathematics currently being done is nonconstructive, why haven't users of mathematics complained? Two facts help to understand this. First, although most mathematicians make no effort to produce constructive results, nevertheless their results often have a very large (and largely hidden) constructive content. It is this hidden constructive content that is useful and applicable, and a major goal of the constructivist program is to make it explicit. The constructive content of a classical theorem, even of its proof, is often sufficient for applications. On the other hand, much current mathematics is hopelessly nonconstructive; it has no numerical meaning and no constructivizations seem possible. Such mathematics is not being applied and will always remain inapplicable. This brings us to the second fact, the time lag between mathematical work and its applications. This can run to centuries, and thus isolates current research from the test of applicability. The mathematics of previous ages is so useful in present applications, that there is a general belief that all current mathematical work will certainly be usable at some time in the future. This belief may be too optimistic.

Numerical meaning

The suggestion that a theorem may be disputed may at first cause some surprise. Mathematics is often presented as a prime example of indisputable knowledge, against which other less certain forms of knowledge are compared. When even mathematical knowledge comes into question, it may seem that all hope is lost. Nevertheless, two points will clarify the situation. First, there is a limited, but crucially important, part of mathematics about which everyone agrees: the integers. Thus, $5 + 7 = 12$ is indeed a good example of indisputable knowledge. The importance of this seemingly small part of mathematics is that constructive mathematics attempts to *build all mathematics upon these solid integers*.

Secondly, serious misunderstanding is caused by differences in interpretation of the meanings of theorems. Superficially, the dispute sounds quite irresolvable. The classicist has a theorem which states that a certain point exists. The constructivist says it does not exist, and even has a counterexample which convinces him that such a point will never be proved to exist. A closer examination shows that the two are using entirely different meanings of "exists." The classicist has indeed proved that the assumption that such a point does not exist leads to a contradiction, and using the principle of excluded middle, has concluded that such a point does exist. Thus by "existence" the classicist merely means "nonexistence is contradictory." On the other hand, the constructivist, when saying that a point exists, means that a procedure has been given by which the point is explicitly constructed.

Thus the dispute is resolved when we consider theorems only in conjunction with their proofs. After all, the statement of a theorem is nothing more than a summary of what has been demonstrated in the proof, using concise (and often misleading) terminology. When both examine the proof, the classicist and constructivist fully agree about what has been proved. Has the controversy vanished into thin air? No! Rather, we have come to the crux of the issue. The crucial question is, What theorems *should* we prove? The classicist says that the theorem just proved settles the problem, and that the constructivist is wasting time with details. The constructivist says that the classicist has a theorem which is splendid as far as it goes, and which points the way to an interesting and useful constructive theorem, but which in itself is incomplete. In the example of the continuous curve, it is not enough to stop when the nonexistence of the point sought is shown to be contradictory; we should continue until we have constructed a point. The construction reduces to the construction of certain integers; it has numerical meaning. Thus the constructivist's answer to the question, "What theorems should we prove?", is given by Bishop's thesis: "Theorems with numerical meaning!"

It is a natural human tendency, a metaphysical impulse, to believe that every meaningful statement must be either true or false. This is understandable, since we are finite beings, and usually speak only of finite matters. But in exploring the mathematical infinite, we might heed Plutarch: "When talking about infinity we are on treacherous ground and we should just try and keep our footing" [**18**]. We can avoid the quicksand of excluded middle and keep to the constructive trail.

A basic metaphysical problem is whether truth exists independently of man. The classical approach to mathematics presumes that truth does exist in itself, perhaps in some Platonic sphere, and we have only to find it. The constructivist believes that mathematics belongs to man, and that we ourselves create it, except for the integers. These integers, which have been created for us, have already blazed the trail for us to follow in our creation of further mathematical truth.

Constructive real numbers

Both geometry and arithmetic are products of man's thought, based on our concepts of space and integer. By emphasizing arithmetic proofs we are asserting that our concept of integer is more reliable. Even in geometry (and related fields, including analysis) we expect more reliable results if the geometric concepts are reduced to arithmetic concepts. We reduce the concept of a point in the plane to the concept of real number; a point has coordinates which are real numbers. Then real numbers are reduced to rational numbers; a real number is generated by a sequence of approximating rational numbers, e.g., the finite portions of an infinite decimal. Finally, a rational number is a ratio of integers. The classicist also reduces points and numbers to integers in this manner. A close examination, however, shows that the classical reduction uses the principle of excluded middle, which leads to properties of points and numbers which are constructively invalid. A striking example of this is the trichotomy of real numbers. This is the classical theorem that says that *for any real number* x, *either* $x < 0$, *or* $x = 0$, *or* $x > 0$. Constructively, it is not true; there is no known general finite procedure which, for each real number x, leads to a proof of one of these three alternatives. Worse, there is a Brouwerian counterexample which shows that we can never expect to find such a procedure.

To see roughly why trichotomy fails constructively, suppose that the real number $x = 0.a_1a_2a_3a_4\ldots$ is given in decimal form. In any finite length of time we can calculate only finitely many of the digits a_i. At some point we might have calculated a million digits and found them all zero. Still, we cannot in general predict whether all the potentially calculable digits will be zero, or whether a nonzero digit might someday appear. Thus we cannot tell whether $x = 0$ or $x > 0$. This example also indicates why the continuous curve theorem discussed above fails constructively. We have no finite process to decide whether the ordinate of a given point is zero or not; we can only calculate approximations to it. Thus, while we can tell whether a point of the curve lies near the axis, we usually cannot tell whether it lies exactly on the axis. In the Sahara problem, the classical camel might tell us to head west if $x = 0$, but to head east if $x > 0$. Since there are infinitely many decimal digits, we might have to perform an infinitely long calculation before we could take even the first step.

The idea of the trichotomy certainly arises intuitively when we draw a line, mark a zero point on it, and look at the picture. However, this geometric picture of the real numbers, though useful, can be misleading. If we look at a line we may be tempted to think only of points which we deliberately mark off on it, and for which we have a preconceived notion about whether they lie to the left, right, or at the zero point. Thus the trichotomy may seem evident, but this naive view does not take into account all real numbers which have already been constructed, or which may be constructed in the future.

The inadmissibility of trichotomy may seem a mighty blow to constructivism, since it seems to be such a fundamental property of the real numbers. On the contrary, our fondness for the trichotomy arises only from its habitual use; it is not essential for constructive analysis. In its place we use other properties of the real numbers which are constructively valid. For instance, given any small positive number ε, it is constructively true that every real number is either less than ε or greater than zero; we have a dichotomy with a small overlap. Such a dichotomy can be used to construct a point on our continuous curve within a distance ε from the axis.

The relation of constructive mathematics to the whole of mathematics

Most classicists view constructive mathematics as a special, rather minor part of mathematics which, for reasons unclear, avoids the use of certain logical principles and methods of proof. Some followers of Brouwer maintain that constructive mathematics forms a separate branch of mathematics, alongside of and distinct from classical mathematics [**11**, p. 4]. The position of modern constructivists differs from each of these. To them it is classical mathematics which is part of the totality of mathematics; this totality is constructive mathematics. The part which is classical, very large today, but very small in the inevitable future, is that part which uses, as an extra hypothesis, the principle of excluded middle. This extra hypothesis has its limited use, as noted above, but its general use is totally unwarranted.

Often constructive mathematics is considered part of formal logic, philosophy, or the foundations of mathematics. Our point of view, however, is that formal logic and philosophy make only an attempt to lay a "foundation" for mathematics. The practice of mathematics requires no foundation other than that it be based on finite constructions which ultimately reduce to the integers.

In the recent history of constructivism, two main questions arise. After Brouwer's critique of classical mathematics, why was nonconstructive mathematics not immediately rejected? And now with the basic part of mathematics already constructivized, and with methods for further constructive progress at hand, why do only a few use these constructive methods?

The answer to the first question is that although Brouwer's critique of classical mathematics was clear, compelling, widely discussed, and accepted as devastating, still no one had a solution to the problem. Brouwer showed the lack of numerical meaning in classical mathematics, but did not convincingly show how mathematics might be done constructively. In fact, Brouwer himself and his followers were convinced that it was not possible to rebuild most mathematics constructively. They thought that most of the beautiful structures of mathematics would be necessarily lost, and they were willing to suffer this loss for the sake of constructibility [**11**, p. 11]. It was not until later, in 1967, that Bishop showed that there need be no loss, but rather a gain of clarity, precision, applicability, and beauty.

The second question is much more difficult. We can only tentatively suggest some possible reasons for the slow growth of constructivism in mathematics since 1967. Some inertia in the university system and the graduate curriculum may be a factor. Connected with this is the heavy burden on students to master an extensive curriculum which does not always leave sufficient time for exploring new ideas. Another factor may be the chilling effect that the political troubles of this century have had on independent thought. The slow growth of constructive mathematics may also be related to the decay of the idea that the purpose of mathematics is to serve the sciences. Finally, many have been diverted from a careful investigation into the meaning of their own work by the existence of those branches of mathematics which attempt to lay a foundation for the rest of mathematics, validate it, and give it meaning. It is not at all clear that this attempt will succeed. If meaning is to be found in a piece of mathematics, *that* is where it will be found. When you prove a theorem, you must yourself show where the meaning lies; you cannot leave this task to others.

Truth is said to reside in a deep well [**14**]. Reach not for the jug of excluded middle to slake a false thirst; strive to draw what is truly needed from the well.

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NOT IN OUR NEXT ISSUE

Editor's note: As a service to our readers, we provide a list below of titles of articles that will not be in our next issue, nor in any issue in the foreseeable future.

Error Analysis of the Quadratic Formula

A Table of Differences of Twin Primes

Classification of the Non-Abelian Groups of Prime Order

Homomorphic Images of Bathtub Rings

Paracompact Parking Spaces

What to Do When Ratios Get Out of Proportion

Efficient Pedaling Techniques for Disjoint Cycles

Descendants of Famous Mathematicians, no. $e^{i\pi}$: The Houston Eulers

—RICHARD LAATSCH
Miami University, Ohio

What Infinite Matrices Can Do

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Must the composition of two matrices be a matrix? Must the inverse of a matrix be a matrix? The answer is *no*. The purpose of this article is to illustrate the failure, for *infinite*-dimensional vector spaces, of familiar facts about finite-dimensional vector spaces and to supply teachers with a stock of examples to show that results proved using associativity cannot be proved without it. The examples also show the necessity of distinguishing between a linear transformation given by a matrix and the matrix itself. The two questions given above are clarified as to meaning and answered in the course of our discussion.

Let V be a vector space of sequences (of real numbers) $x=\{x_n\}$. When the sequences in V satisfy $x_j=0$ for all $j\geqslant N$, then V can be regarded as a finite-dimensional vector space. We say that a map T is a **matrix map** if V is the domain of T and there exists a matrix A such that, for all $x\in V$,

$$T(x)=\{(Ax)_n\},\quad \text{where } (Ax)_n=\sum_{k=1}^{\infty} a_{nk}x_k.$$

When T is a matrix map, we will often identify T and its defining matrix A, and write $T=A$. When V is finite-dimensional, then every linear map from V to another finite-dimensional vector space is a matrix map. However, when V is infinite-dimensional, this fails, as the following example shows.

EXAMPLE 1. Let c be the space of convergent sequences and define $T\colon c\to c$ by $T(x)=(\lim x,0,0,0,\dots)$. Clearly T is a linear map, but *T is not a matrix map*. To see this, suppose that a matrix A exists with $T(x)=\{(Ax)_n\}$. If we take $x=\delta^m$ (i.e., $x_m=1$, $x_k=0$ for $k\neq m$) then $0=T(x)=\{a_{nm}\}$, the mth column of A. Hence A is the zero matrix. But for the constant sequence $\{1\}$, $T(\{1\})=\delta^1\neq 0$, a contradiction.

However, *T is the composition of two matrix maps B: $c\to cs$, A: $cs\to c$*, where cs is the space of sequences y such that the infinite series $\sum y_n$ is convergent. The matrix B is given by

$$B=\begin{pmatrix} 1 & 0 & 0 & \cdots \\ -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

so that $Bx=(x_1,x_2-x_1,x_3-x_2,\dots)$ for $x\in c$, and the matrix A is given by

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

so that $Ay=(\sum y_k,0,0,0,\dots)$ for $y\in cs$. Then for $x\in c$,

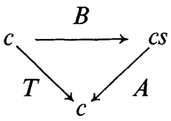
$$A(Bx)=(x_1+(x_2-x_1)+(x_3-x_2)+\cdots,0,0,0,\dots)=(\lim x,0,0,0,\dots),$$

since

$$\sum_{k=1}^{m}(x_k-x_{k-1})=x_m\to\lim x.$$

If we denote by $A\circ B$ the map defined by $(A\circ B)(x)=A(Bx)$, then $A\circ B=T$.

The maps A, B, T in EXAMPLE 1 operate as in this diagram:



I do not know any example in which the domain and range of T, A, B are all the same vector space.

It is easy to check that the *matrix* product AB is the zero matrix in EXAMPLE 1. Thus $A \circ B \neq AB$ since the left-hand side is T. This just says that $(AB)x \neq A(Bx)$ for some x (in this case $x = \{1\}$), i.e., associativity of multiplication fails. This failure is essential to the example, for if $(AB)x = A(Bx)$ for all x, then the map given by $A \circ B$ is certainly given by a matrix, namely AB. A little more is true:

THEOREM. *Let A, B be matrices. If $A \circ B$ is a matrix map, it is AB.*

Let $C = A \circ B$. Then c_{nk} is the nth term in the kth column of C:

$$c_{nk} = (C\delta^k)_n = (A(B\delta^k))_n = ((AB)\delta^k)_n = (AB)_{nk}.$$

The last equality is a result of the easily verified fact that $A(Bx) = (AB)x$ for all x such that $x_n = 0$ for sufficiently large n.

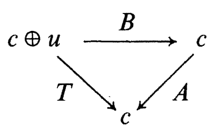
The phenomenon in EXAMPLE 1 is not restricted to singular matrices.

EXAMPLE 2. Let

$$B = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} & \cdots \\ 0 & 0 & 1 & \frac{1}{2} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix},$$

so that $AB = BA = I$. Let $u = \{2^n\}$ so that $Bu = 0$. Then $u = (AB)u \neq 0$ and $A(Bu) = 0$ so by the Theorem, $A \circ B$ is not a matrix map since it is not equal to AB.

The map diagram for this example is



where $c \oplus u = \{x + mu\colon\ x \in c,\ m$ a real number$\}$, which is close to finding an example in which the domain and the range of A, B, T are all the same space. Perhaps it is impossible to find such an example, that is, perhaps if AB, $A(Bx)$ and Ax all exist, it may follow that $A(Bx) = (AB)x$. If this is false, i.e., if such an example exists, it is not hard to see that an example can be found with the maps operating on and to an *FK* space. (An *FK* space is a sequence space with a complete linear metric such that each map $x \to x_n$ is continuous. By methods shown in [**4**] it can be shown that $\{x\colon\ A(Bx) = (AB)x\}$ is an *FK* space, A, B being fixed matrices.)

The set of $n \times n$ matrices is an algebra, that is, a vector space X together with an associative multiplication such that $m(xy) = (mx)y = x(my)$ for any $x, y \in X$ and scalar m. It is a familiar fact that if an element b of an algebra has a unique left inverse a, then a is also its unique right inverse. Associativity of multiplication in the algebra is essential, as the proof makes clear: Suppose b has a unique left inverse a, i.e., $ab = 1$ (the algebra identity). Then $(ba - 1 + a)b = b(ab) - b + ab = 1$ so, by uniqueness, $ba - 1 + a = a$, $ba = 1$ so $a = b^{-1}$. In contrast with this, EXAMPLE 2 shows two infinite-dimensional matrices, B, A, where B has *A as its unique left inverse* ($AB = I$) but *B has many right inverses*: just add a multiple of u to any column of A. Going even further, it is possible for an infinite dimensional matrix to have a unique left inverse and no right inverse at all. For convenience in a subsequent application we show an equivalent example, a matrix A *with a unique right inverse and no left inverse.*

EXAMPLE 3. Let

$$A=\begin{pmatrix}1&1&1&1&1&\cdots\\1&0&1&1&1&\cdots\\1&0&0&1&1&\cdots\\1&0&0&0&1&\cdots\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdots\end{pmatrix},\qquad B=\begin{pmatrix}0&0&0&0&0&\cdots\\1&-1&0&0&0&\cdots\\0&1&-1&0&0&\cdots\\0&0&1&-1&0&\cdots\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdots\end{pmatrix}.$$

Then $AB=I$ and it is trivial to check that $Ax=0$ implies $x=0$, so B is the unique right inverse of A. To see that A has no left inverse matrix, suppose r is the first row of such a matrix. Then $\Sigma r_i=1$ but $\Sigma_{i=1}^n r_i=0$ for $n=1,2,\dots$. This is impossible.

The matrix A in EXAMPLE 3, considered as a matrix map from cs to c, *has an inverse map which is not a matrix map*. To see what the inverse map M is, let $x\in c$, and $y=M(x)\in cs$. Then $x=Ay$ so that $x_n=y_1+\Sigma_{k=n+1}^{\infty}y_k$ from which $y_1=\lim x$, $y_{n+1}=x_n-x_{n+1}$, i.e., $y=T(x)+Bx$, where T is defined in EXAMPLE 1 and B is defined in EXAMPLE 3. Thus $M=T+B$, which (using an argument as in EXAMPLE 1) is not a matrix map. The matrix A in EXAMPLE 3 does not have an inverse matrix.

An example is given in [3], p. 407, Theorem 11, of *a matrix which has an inverse matrix and an inverse map which are different*, i.e., the inverse map is not given by the inverse matrix! This illustrates a curious possibility for our Theorem: let T be the map given by the matrix A such that T^{-1} is not given by A^{-1}; $T\circ T^{-1}$ *is* given by the matrix $AA^{-1}=I$.

In [**1**], pp. 245–247, are shown several symmetric row-finite matrices *each of which has infinitely many two-sided inverse matrices*. The inverses shown are also symmetric. Let A be such a matrix, i.e., A is symmetric, row-finite, has a symmetric two-sided inverse and a symmetric nonzero matrix Z such that $ZA=AZ=0$ (Z is the difference between two inverses.) It is proved in [**2**], Theorem 3c, that A has a row-finite right inverse R. Paradoxically R *cannot be symmetric* for if it were we would have $RA=I$ and so $0=R(AZ)=(RA)Z=Z$.

A talk given to the Eastern Pennsylvania and Delaware Section of the MAA on November 19, 1983.

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Another Way to Discover that $\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{n2^{n+1}n!}=\ln 2$

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The article [1] caught my eye. That identity looks familiar, I thought, and then I realized that I had recently discovered the same identity by a method which is also somewhat serendipitous. This involved consideration of the following problem in probability.

The World Series is played as the best of seven games. Suppose that the better team (say A) has a fixed probability p ($>1/2$) of winning any particular game and that, more generally, a series of $2n+1$ games is played. What is the probability P_n that A wins the series?

First, it is intuitively plausible that if enough games are played, the better team is almost sure to come out on top, i.e., $\lim_{n\to\infty}P_n=1$. This can be established using Tchebychev's well-known

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First, it is intuitively plausible that if enough games are played, the better team is almost sure to come out on top, i.e., $\lim_{n\to\infty}P_n=1$. This can be established using Tchebychev's well-known inequality [2]. Let X denote the number of games A wins. (Here it is convenient and harmless to suppose that all $2n+1$ games are played, although in practice, of course, the series would be stopped as soon as one team reached $n+1$ wins.) Then X has the binomial distribution: $b(2n+1, p)$. Now $P_n = \Pr(X \geq n+1) = 1 - \Pr(X \leq n)$, and we find that (recall that the mean of X is $(2n+1)p$)

$$\begin{aligned} \Pr(X \leq n) &\leq \Pr(|X-(2n+1)p| \geq (2n+1)p - n) \\ &\leq \frac{\operatorname{Var} X}{((2n+1)p-n)^2} \qquad \text{(by Tchebychev)} \\ &= \frac{npq}{n^2\left(2p+\frac{1}{n}p-1\right)^2} \\ &\leq \frac{pq}{n(2p-1)^2} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This establishes that $\lim_{n\to\infty} P_n = 1$. Next let us examine the individual P_n. We observe that $P_n = \Pr(X \geq n+1)$ is a sum of binomial probabilities (with $q \equiv 1-p$):

$$P_n = p^{n+1}\left[\binom{n+0}{0} + \binom{n+1}{1}q + \binom{n+2}{2}q^2 + \cdots + \binom{n+n}{n}q^n\right]. \tag{1}$$

Using (1) to compute $P_n - P_{n-1}$ and the basic relation

$$\binom{n+k}{k} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k}$$

to simplify the result, we find

$$P_n - P_{n-1} = p^n q^n \binom{2n-1}{n}(1-2q). \tag{2}$$

(Note this implies that P_n increases with n, which is not surprising. Indeed, it was an effort to establish the positivity of $P_n - P_{n-1}$ which motivated (2).) Summing (2) from 1 to n yields

$$P_n = p + (1-2q)\left(pq + \binom{3}{1}(pq)^2 + \binom{5}{2}(pq)^3 + \cdots + \binom{2n-1}{n}(pq)^n\right). \tag{3}$$

We now take the limit as $n \to \infty$ of (3), then divide by q and let $x = 4pq$ (which implies $0 \leq x < 1$ and $p = (1+\sqrt{1-x})/2$) to obtain

$$\frac{2}{\sqrt{1-x}+1-x} = 1 + \binom{3}{1}\frac{x}{4} + \binom{5}{2}\frac{x^2}{4^2} + \cdots, \qquad 0 \leq x < 1. \tag{4}$$

Integrating (4) term by term (no problem for a series expansion) gives the interesting Maclaurin series (at least for $0 \leq x < 1$)

$$\ln(1+\sqrt{1-x}) = \ln 2 - \frac{x}{4} - \frac{1}{2}\binom{3}{1}\frac{x^2}{4^2} - \frac{1}{3}\binom{5}{2}\frac{x^3}{4^3} - \cdots. \tag{5}$$

The series on the right in (5) converges at $x=1$ (Raabe's Test), hence converges uniformly on $[-1,1]$ (Weierstrass M-Test); hence, its sum is continuous on $[-1,1]$. This permits us to put $x=1$ in (5) to obtain the identity of the title.

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A King's Tour of the Chessboard

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The king in chess can move one square in any direction, horizontally, vertically, or diagonally. With no restrictions, a king's tour of the chessboard, visiting each square once and only once, is easy. (Conducting a knight's tour is a far more challenging puzzle that has received much study [**1**, pp. 257–66].) Now add the restriction that the king (after the first move) can only move to a square which touches an *even* number of squares which have already been visited. This problem is much harder; in fact, *it is impossible*.

The impossibility follows from a simple parity argument in the case of the standard 8×8 chessboard, or any chessboard with both dimensions even or both dimensions odd (except the trivial 1×1 board.) Two squares of the board are called a **neighbor pair** if they touch, either at their corners or along their sides. A neighbor pair is **completed** if both its squares have been visited by the king, including the case in which the king is still sitting on one of the two squares. The rules of the tour require the king to complete one neighbor pair on its first move and an even number of neighbor pairs on each subsequent move. Thus the total number of completed neighbor pairs (call it N) at any stage of the tour is odd. On a board with m rows and n columns, there are $m(n-1)$ horizontal neighbor pairs, $n(m-1)$ vertical neighbor pairs, and $2(m-1)(n-1)$ diagonal neighbor pairs. If the king can finish the tour, he will complete $N = 2mn - (m+n) + 2(m-1)(n-1)$ neighbor pairs. Since this total must be odd, $m+n$ must be odd, and thus m and n must have opposite parity.

In case the dimensions m, n of the board do not have the same parity, we need a different argument to prove a tour is impossible. We will keep track of another quantity that is not so directly related to the rules of the game. Four squares of the board that meet at a point (thus forming a 2×2 square) will be called a **foursome**. The foursomes are in one-to-one correspondence with internal vertices of an $m \times n$ board, and so there are $(m-1)(n-1)$ of them. A foursome is **completed** when the king has visited its four squares, including the case where the king is still sitting on the fourth square. We will let F stand for the number of foursomes that the king has completed at a given stage of the tour.

We shall use the letter S to denote the total number of squares that the king has visited. (This is one more than the number of moves the king has made.) In analyzing many tours, we discovered that the quantity

$$I = S + F - \frac{N+1}{2} \tag{1}$$

has the property that it cannot increase (with one exception) as the tour progresses.

Lemma. *Except for the first and last move, the value of I either decreases or stays the same.*

Proof. We will say that on a given move, the quantities I, S, F and N change by amounts ΔI, ΔS, ΔF and ΔN. The definition of a move gives us $\Delta S = 1$ and the "even rule" gives us $\Delta N = 2$, 4, 6 or 8 (the rules of chess eliminate $\Delta N = 0$). The geometry of the board gives us $\Delta F = 0, 1, 2, 3$ or 4 and the definition of I in (1) gives us $\Delta I = \Delta S + \Delta F - \Delta N/2 = 1 + \Delta F - \Delta N/2$.

We want to show that, except for the last move, $\Delta I \leqslant 0$. We do this by looking at the possible values of ΔF in turn. If $\Delta F = 0$, then $\Delta I \leqslant 0$ since $\Delta N \geqslant 2$. If $\Delta F = 1$, then the latest move completes at least three neighbor pairs, and so by the "even rule" $\Delta N \geqslant 4$ and $\Delta I \leqslant 0$. If $\Delta F = 2$ then the two completed foursomes overlap at least in the latest square visited, and at most in that square and one other square. In the former case, the king is adjacent to at least six filled squares, and in the latter case, to at least five filled squares. By the even rule, $\Delta N \geqslant 6$ in either case, and

FIGURE 1

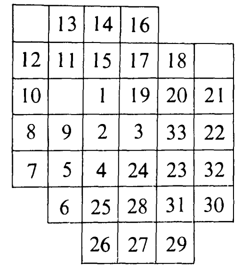

FIGURE 2

this makes $\Delta I \leqslant 0$. If $\Delta F \geqslant 3$, then $\Delta N \geqslant 7$, implying $\Delta N = 8$, $\Delta F = 4$ and $\Delta I = 1$. But after such a move, the king is sitting on a square that is surrounded by eight squares he has already visited, and so his tour is over.

The first two moves of a tour can only occur in the two equivalent ways shown in FIGURE 1. These create a configuration in which $I = 1$. A consequence of the Lemma is that no subsequent value of I can ever be larger than 2. Using the Lemma, we can complete the proof of our assertion made at the outset of this note.

THEOREM. *A king's tour under the stated conditions cannot be completed on any rectangular chessboard except a* 1×1 *or* 1×2.

Proof. Recall that on an $m \times n$ chessboard, $S = mn$, $F = (m-1)(n-1)$, $N = m(n-1) + n(m-1) + 2(m-1)(n-1)$, and so $I = (1/2)(m + n - 1)$. If $m + n \geqslant 7$, then $I \geqslant 3$ and the tour cannot be completed. If $m + n = 5$, we have a 1×4 or 2×3 chessboard. Here, the value of I is 2, but it is not possible to make a move with $\Delta I = 1$ because, from the proof of the Lemma, such a move must complete eight neighbor pairs. But no square on the board has eight neighbors. If $m + n = 3$, we have a 1×2 board, on which the tour *can* be completed because the king is allowed to violate the even rule on the first move.

The Theorem leaves open the question of what fragments of chessboards *can* be toured. This problem, without the chess king metaphor, was originally proposed by Sid Sackson in [**2**] for the fragment shown in FIGURE 2. Since Sackson's board has $S = 36$, $N = 111$, $F = 24$ and hence $I = 4$, a tour of his board is impossible. The tour indicated in FIGURE 2, which visits 33 of the 36 squares, was found using a computer program written by Mark Meyerson. The method of this paper does not rule out a tour of 34 squares, omitting two isolated internal squares. A computer search indicates, however, that no such tour is possible.

Sackson also proposed the rule that each square visited must be adjacent to an *odd* number of squares already visited. He showed that his board can be toured using this rule, but our computer searches have turned up no instances of rectangular boards other than $1 \times n$ boards that can be toured. The quantity I in (1) rises and falls uncontrollably under these moves. (Necessarily: if the value of I only decreased under odd as well as even moves, then an unrestricted king's tour of the chessboard would be impossible, which is absurd.)

Note that we use the chess rule that a king must move to an adjacent square only when we argue that a move with $\Delta N = 8$ must end the tour and when we state that $\Delta N \neq 0$. Such moves are illegal using the "odd" rule; a complete king's tour of a rectangular chessboard must end on the edge of the board under this rule. This distinction may account for our inability thus far to come up with a quantity comparable to I for the odd rule.

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"Can You Ski?"

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While tagging along on a school ski trip, Dr. Macal was asked by a spritely freshman who sat next to her: "Can you ski?" It was clearly a surprise to the freshman to see Dr. Macal in skiing attire rather than in her customarily trendless clothing. Dr. Macal gathered her thoughts a bit.
Macal: It depends on what you expect it to mean. Clearly there are several levels of ability. Suppose we call the set of all the people on this bus B. If M is the universe of all the people who are associated with our school, then for each student s in M we may define a two-valued membership function.

$$b(s)=1 \quad \text{if } s\in B, \qquad b(s)=0 \quad \text{if } s\notin B.$$

Suppose we ask anyone in the lobby of the College Union whether or not they are coming with us on this trip. What responses do we get?
Linda: Either yes or no.
Macal: The set B is clearly defined. Mathematicians call such a set crisp (or sharp) because not all sets can be so clearly defined. Suppose you are only interested in having an answer to your original question about the skiing ability of each member of B. Then for each s in B you want to associate a value which depends on s, that is, a value which somehow measures the proficiency of the skiing ability of s. This will define a different function b, having domain B, and the set B can be called the support set of your inquiry. If we decide to measure skiing ability on a scale of 0 to 1, then the values $b(s)$ all lie in the interval $[0,1]$ with $b(s)=0$ if s cannot ski, $b(s)=1$ if s is a pro, and $0<b(s)<1$ otherwise. Of course, we could use a different range, such as 0 to 10.
Linda: That does not take care of the fact that there are *two* kinds of skiing abilities: cross-country and downhill.
Macal: Terrific! Then we can define another function, c, on the set C of all people who ski cross-country with the same range of values, from 0 to 1. This way, c measures the ability to ski cross-country. Similarly, for the set D of all people who can ski downhill, a function d will also assign values to members of D from the interval $[0,1]$. These functions, b, c, and d, by the way, are called membership functions for the sets B, C, and D, respectively.
Linda: These sets make sense because there is far more information associated with each member of the set than in the crisp case. I wonder what the Venn diagrams are going to look like! Would a shaded circle with high membership values (darker shade) inside and low membership (lighter shade) towards the outside do?
Macal: I guess these sets are fuzzy! But don't you think that a circle with various degrees of shading would be difficult to work with?
Linda: Absolutely!

Three years after that skiing trip, Linda graduated with Honors in Mathematics after having completed two semester's research on an application of fuzzy sets. Her written thesis and her oral defense were evaluated by a panel of five people from the academic and industrial environment. Her thesis includes a model for the selection of a type of electric plant [**3**]. Who said that the seed of learning may germinate anywhere? The dialogue (or a reasonable facsimile of it) actually took place, and is reported as an example of teaching without the formal setting of the classroom.

The most attractive feature of fuzzy sets is that it affords an applicable rendition of the notion of belonging to complex situations for which "belonging" cannot be defined sharply. It is precisely this feature that has made fuzzy sets grow in popularity among practitioners since the publication of L. Zadeh's seminal paper in 1965 [**22**]. There is some reluctance in the mathematical community to accept "fuzzy" mathematics. It may be caused by its funny name. This reluctance is puzzling because fuzzy set theory proposes an interesting extension of the concept of set. Thus the main purpose of this presentation is to introduce the basic concepts of the mathematics of fuzzy sets and some of their properties. At the same time, care is used to point out how fuzzy sets generalize the concept of set which is learned in classical logic and set theory courses. This way, the reader will realize that there is a great deal of difference between doing "fuzzy mathematics" and doing "mathematics fuzzily."

What is a fuzzy set?

In this section, the concept of fuzzy set is defined. Some basic operations are discussed. Throughout, interesting issues that are worth pursuing are singled out.

For simplicity, assume that B is a collection of a finite number of elements:

$$B=\{s_1,\dots,s_n\}.$$

Recall that the set indicator function i_B assigns to each element of B the number 1, and assigns 0 to any element not in B. The set B, together with this set membership function i_B is called a **crisp**, or **sharp** set. The graph of i_B is the set of ordered pairs

$$\{(s_1,1),\dots,(s_n,1)\}.$$

This graph is contained in $B\times[0,1]$. The set $B\times[0,1]$ also contains sets of ordered pairs $(s,c(s))$ where c is any single-valued mapping from B into the interval $[0,1]$.

Any set of such pairs $(s,c(s))$ is called a **fuzzy set** with **support set** B and **membership function** c. Such a fuzzy set will be denoted as B_f (f is for fuzzy!); the particular membership function for a given fuzzy set will always be specified and denoted by a lower case letter. If the membership function is i_B, the set B_f is crisp, in which case B_f is identifiable with B.

For example, if B is the set of people in a bus then C_f could denote the fuzzy set of people in the bus who ski cross-country and who are not necessarily pros. The membership function would be c, as described in the dialogue. Another example of a fuzzy set with the same support B is the set D_f whose membership function is the single-valued mapping that measures the ability of skiing downhill on a scale of 0 to 1. Equality between the set C_f and D_f is not possible unless each individual in the bus skis with the same ability in both cross-country and downhill. Thus equality for the fuzzy sets C_f and D_f having the same support set B and membership functions c and d, respectively, is defined by

$$C_f=D_f \quad \text{if and only if } c(s)=d(s) \quad \text{for all } s\in B.$$

This just says that two fuzzy sets are equal if the graphs of their membership functions are the same. Since, in the example, c and d assign the ability to ski on a scale from 0 to 1, "less ability" for cross-country skiing than for downhill is translated as $c(s)\leqslant d(s)$ for an element s of B. If each person on the bus is less good at cross-country than at downhill skiing then it is reasonable to say that the set C_f is contained in D_f. Thus inclusion for fuzzy sets is defined by

$$C_f\subseteq D_f \quad \text{if and only if} \quad c(s)\leqslant d(s) \quad \text{for all } s\in B.$$

In ordinary set theory, the union of two sets is the smallest set containing both sets. To maintain this property, the membership function for $C_f\cup D_f$ should be the function $\max\{c,d\}$. Recall the examples described in the dialogue: the union must describe how good each s is at either of the two ways of skiing. An element s belongs to either set with the larger of its degree of membership in each set. An element s belongs to both of the two sets with the lesser of its degree of membership in each set—so the $\min\{c(s),d(s)\}$ is defined to be its degree of membership in the intersection. The definition of union and of intersection of two fuzzy sets follows.

$$C_f\cup D_f=W_f \quad \text{if and only if } w(s)=\max\{c(s),d(s)\} \quad \text{for all } s\in B,$$

$$C_f\cap D_f=V_f \quad \text{if and only if } v(s)=\min\{c(s),d(s)\} \quad \text{for all } s\in B.$$

The operation of complementation for fuzzy sets is the most obvious to define. If an individual skis downhill with ability $d(s)$, then $1-d(s)$ represents the skill that is lacking, the amount

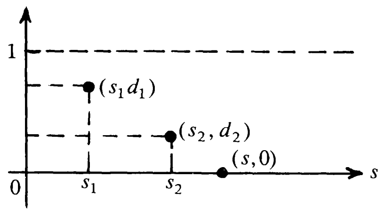


FIGURE 1

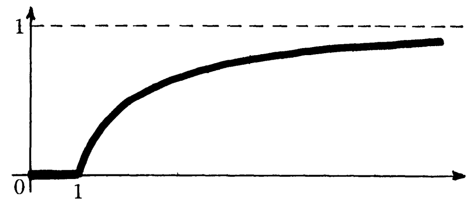


FIGURE 2. **The graph of** $b(x)=\begin{cases} 0 & \text{if } x \leqslant 1 \\ 1-x^{-1} & \text{if } x>1. \end{cases}$

needed for improvement. Thus the complement of D_f is the set of pairs $(s, d'(s))$ with $d' = 1 - d$.

$$D_f' = \{(s, d'(s)) \mid d'(s) = 1 - d(s), \quad s \in B\}.$$

Does the fact that the set B of the example is finite play a basic role in the definition of fuzzy sets? Since the set B is finite, the fuzzy set D_f can be represented as a finite set of points in the plane by arranging the elements of B in linear order along a horizontal axis and plotting the corresponding membership values with respect to a vertical axis (see FIGURE 1). Notice that any individual s in B who cannot ski downhill can always be included in D_f by using the pair $(s,0)$. If the set B is not finite but is still *linearly ordered*, then a fuzzy set can be defined by some membership function. For example, the set of real numbers x that are "much greater than 1" forms a fuzzy set with support set $\mathbb{R}$. The membership function must equal 0 for $x \leqslant 1$, and its value must increase to 1 as x increases and $x > 1$. According to the context in which the statement "much greater than 1" is made, the selection of such a membership function may be subject to further restrictions. However, if there are no additional requirements, then one admissible choice for such a function is

$$b(x) = \begin{cases} 0 & \text{if } x \leqslant 1; \\ (x-1)/x & \text{if } x > 1. \end{cases}$$

The graph of $b(x)$, which lets us visualize this fuzzy set, is given in FIGURE 2. As an example of a different membership function for the support set $\mathbb{R}$, consider the fuzzy set of real numbers x that are positive and "close to 5." If no other information is given, then the sketch of the graph of the membership function of FIGURE 3 is an admissible choice. Functions whose graphs look like FIGURE 2 and FIGURE 3, respectively, are sometimes referred to as the "S" function type and the "π" function type [7, p. 28]. Is there a mathematical reason for requesting that the support set B be linearly ordered? In this introduction, it makes sense because the membership function defines the degree to which an element belongs to a set. Mathematically, it is not necessary. It suffices to say that if the set B is partially ordered, a lattice structure is used [**8**, p. 84 ff].

An interesting graphical representation of the set operations is achieved when two fuzzy sets, C_f and D_f, share the same support B which equals a closed interval $[a, b]$. TABLE 1 shows the graphs that result from the definitions of the membership functions for the sets $C_f \cup D_f$, $C_f \cap D_f$ and D_f'. The meaning of $C_f \subseteq D_f$ is also shown. Notice that if $B = [a, b]$ then $B \times \{0,1\}$ consists of two parallel horizontal line segments on $y = 0$ and $y = 1$. The set $B \times [0,1]$ is the entire rectangle with

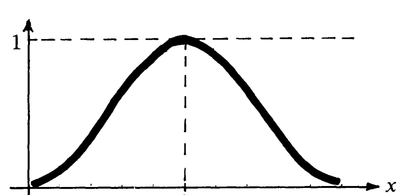
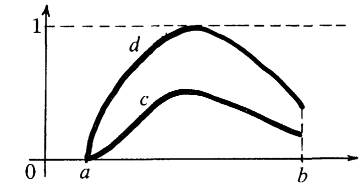
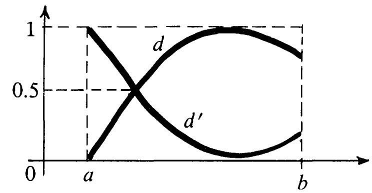


FIGURE 3. **The graph of an admissible membership function for the fuzzy set of all numbers that are "close to 5."**

$C_f \subseteq D_f$



D_f'



$C_f \cup D_f = W_f$

$C_f \cap D_f = V_f$

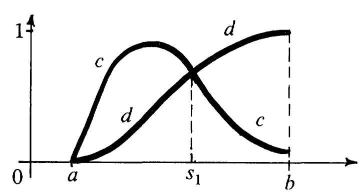


TABLE 1. **In the above illustrations, the fuzzy sets C_f and D_f have support functions c and d, respectively. The support function w of W_f has graph $\{(s, c(s)) | s \leqslant s_1\} \cup \{(s, d(s)) | s \geqslant s_1\}$. The support function v of V_f has graph $\{(s, d(s)) | s \leqslant s_1\} \cup \{(s, c(s)) | s \geqslant s_1\}$.**

base $[a, b]$. Kaufmann refers to the shaded region under the sketch of the membership function over $[a, b]$ as a Venn diagram [9, p. 13]. It is correct because of the definition of containment: if (s, d) belongs to D_f then any (s, g) with $g \leqslant d$ belongs to some fuzzy set contained in D_f. See the first illustration in TABLE 1.

Other set operations on fuzzy sets exist. For example, there is the algebraic product and the direct sum [7, pp. 30–39]. The **algebraic product** of a set C_f with a set D_f, denoted $C_f D_f$, is sometimes used as an alternative definition for the intersection of two fuzzy sets: an element s belongs to the algebraic product with membership defined by the product $c(s)d(s)$. In [3], it is suggested that the algebraic product gives more importance to the largest membership value. Its use is recommended when the membership functions have been determined very accurately. The **direct sum** of two fuzzy sets is denoted $C_f \oplus D_f$, and the degree of membership of an element s in the direct sum is given by $c(s) + d(s) - c(s)d(s)$. It is straightforward to show from the above definitions that

$$C_f D_f \subseteq C_f \cap D_f \subseteq C_f \cup D_f \subseteq C_f \oplus D_f.$$

Notice that if the degree of membership is restricted to the 2-element set $\{0, 1\}$, instead of the continuous interval $[0, 1]$, then all the above definitions yield the same results as the definitions we learn in the classical sense. This is one of the reasons why it is often stated that fuzzy set theory is a generalization of classical set theory. There are other reasons; these will be discussed in the next section.

It seems appropriate to discuss here an area which is worth further study. What happens if the membership function does not depend solely on the elements of the support set? Often, the ability to ski downhill is not constant throughout a day of skiing. In other words, the membership function depends on the individual s and on the time t. Suppose that Linda's ability increases

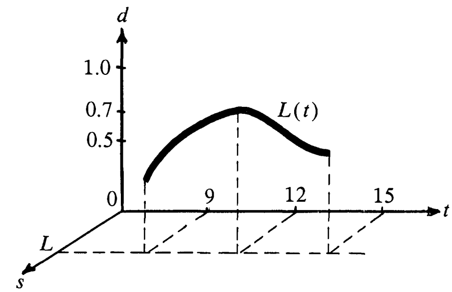


FIGURE 4

from a minimum value .5 at 9:00 a.m. to a peak performance around noon, and then it decreases to .7 around 3:00 p.m. The graph $L(t)$ of Linda's ability, rather than a line (a constant function) will look something like the one in FIGURE 4. Of course, the shape of this "ability graph" is not the same for all individuals. A beginner may start the day very poorly and then steadily improve. Oddly, the extension of the concept of fuzzy sets to a time dependent setting has received very little notice [**11**].

Properties of fuzzy sets

The emphasis on the treatment of a fuzzy set via functions often detracts from the set perspective. Are there properties that hold for fuzzy sets and that are similar to those in set theory? It will be shown that while some properties from set theory continue to hold, others are lost, with interesting consequences.

To begin, since

$$\max\{c, c\} = c = \min\{c, c\},$$

the idempotent property holds for union and intersection of fuzzy sets. It also easily follows that the union and the intersection of fuzzy sets is commutative. The validity of the associative law

$$C_f \cup (D_f \cup G_f) = (C_f \cup D_f) \cup G_f$$

can be verified by listing the six possible cases that hold in comparing membership functions and giving a proof for each case. The six cases are:

$$c \geqslant d \geqslant g, \quad c \geqslant g \geqslant d, \quad d \geqslant c \geqslant g,$$
$$d \geqslant g \geqslant c, \quad g \geqslant c \geqslant d, \quad g \geqslant d \geqslant c.$$

The proof is given for the first case only, because it is the same for all the others. Since $c \geqslant d \geqslant g$ it follows that

$$\max\{d(s), g(s)\} = d(s) \quad \text{and} \max\{c(s), d(s)\} = c(s),$$
$$\max\{c(s), \max\{d(s), g(s)\}\} = \max\{\max\{c(s), d(s)\}, g(s)\}.$$

Thus the associativity law holds. It is also left to the reader to show that since the intersection involves the min operator, the associativity law

$$C_f \cap (D_f \cap G_f) = (C_f \cap D_f) \cap G_f$$

holds for the intersection too.

Again a proof by cases will show that the usual distributive laws for union and intersection hold. Assume (as before) that the first case is given by $c(s) \geqslant d(s) \geqslant g(s)$. Since

$$\min\{c(s), \max\{d(s), g(s)\}\} = d(s)$$

and

$$\max\{\min\{c(s), d(s)\}, \min\{c(s), g(s)\}\} = d(s),$$

it follows that

$$C_f \cap (D_f \cup G_f) = (C_f \cap D_f) \cup (C_f \cap G_f).$$

Since

$$\max\{c(s), \min\{d(s), g(s)\}\} = c(s)$$

and

$$\min\{\max\{c(s), d(s)\}, \max\{c(s), g(s)\}\} = c(s),$$

it also follows that

$$C_f \cup (D_f \cap G_f) = (C_f \cup D_f) \cap (C_f \cup G_f).$$

The arguments for the other five cases are similar to the one given above.

To surmise that most other set theory properties continue to hold would prove wrong. Recall the example of Linda's ability to ski. Only when her style is perfect ($d = 1$) is there no need for improvement ($d' = 0$). On the other hand, a neophyte ($d = 0$) has a lot to learn ($d' = 1$). If G_f is the intersection of a set D_f with its complement D_f', then the degree of membership $g(s)$ of an element s of G_f is

$$g(s) = \min\{d(s), d'(s)\} = \min\{d(s), 1 - d(s)\}.$$

If $0 < d(s) < 1$, then $g(s) \neq 0$. In other words, the intersection $G_f = D_f \cap D_f'$ need not be empty. An interesting interpretation of this fact follows. Let $d(s)$ describe the **ability curve**. Whenever the values of the membership function $g(s)$ for the intersection cluster around .5, mediocre is the attribute for the collective performance of the elements in the set B. Indeed, the highest value of $g(s)$ is realized when $\min\{d(s), d'(s)\} = 1/2$, i.e., $d(s) = d'(s)$, and $D_f = D_f'$. It can be said that a lack of distinction between a fuzzy set and its complement is an undesirable feature. It is interesting to mention that a measure of the fuzziness of a set B_f is possible because it may be computed from the values $d(s), d'(s)$ [**21**, p. 224].

What is a **fuzzy universe**? The set $B \times [0,1]$ could be chosen as the universe, but the second figure of TABLE 1 shows that *all* fuzzy sets with support B are contained in $B \times [0,1]$ (it follows from the definition of Venn diagram according to Kaufmann [**9**]). A **relative universe** is defined as the smallest set containing both the fuzzy set and its complement.

$$U_D = D_f \cup D_f'.$$

Notice that the relative universe is D_f itself when D_f is crisp.

DeMorgan's Laws do hold. For example, to prove that

$$(C_f \cap D_f)' = C_f' \cup D_f',$$

it must be shown that the value of the membership function for an arbitrary element s is the same for the sets on either side of the equation. Assume that $c(s) \geqslant d(s)$. Then $1 - c(s) \leqslant 1 - d(s)$, and so $\max\{1 - c(s), 1 - d(s)\} = 1 - d(s)$ is the value of the membership function of the set on the right side of the equation. Since $\min\{c(s), d(s)\} = d(s)$, the membership function of the set on the left-hand side also equals $1 - d(s)$. The law

$$(C_f \cup D_f)' = C_f' \cap D_f'$$

is proven in a similar way.

The reader may wish to investigate what properties hold for the algebraic product and the direct sum.

It seems appropriate to end this introduction with a topic that combines many of the above facts on fuzzy sets. Thus the remainder of this section is devoted to the notion of convexity. This concept is needed to define fuzzy numbers [**5**], and is a particularly useful notion in applications which make use of partition spaces for pattern classification purposes [**2**, p. 29]. For simplicity, the

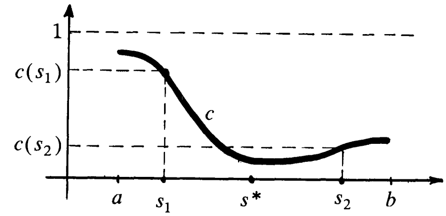


FIGURE 5. **The fuzzy set with membership function c is not convex.**

discussion is once again restricted to the case in which the support set B is an interval of the real line [**22**], [**8**, p. 58].

Recall that a function $c(s)$ defined on an open interval (a, b) is called a **convex function** (also called concave upward) if the condition

$$c(ts_1 + (1-t)s_2) \leqslant tc(s_1) + (1-t)c(s_2) \tag{1}$$

holds for all $0 \leqslant t \leqslant 1$ and for all s_1, s_2 in (a, b). Geometrically, condition (1) implies that each point on the chord joining any two points of the graph of a convex function lies above the graph [**18**, p. 108]. Consider the region A enclosed by the horizontal axis, the two vertical lines at $s = a$, $s = b$, and by the graph of $c(s)$. Clearly, A is not a convex set unless $c(s)$ is a constant function [**18**, p. 203]. However, according to Kaufmann, if C_f is the fuzzy set with support $[a, b]$ and membership function $c(s)$, then A is the Venn diagram of C_f.

A fuzzy set C_f is said to be **convex** if the condition

$$c(ts_1 + (1-t)s_2) \geqslant \min\{c(s_1), c(s_2)\} \tag{2}$$

holds for all $0 \leqslant t \leqslant 1$ and for all s_1, s_2 in $[a, b]$. To justify such a definition, recall the example in the dialogue. If s_1 and s_2 are two individuals with cross-country skiing ability given by $c(s_1), c(s_2)$, respectively, then convexity holds if no one "between" s_1 and s_2 has an ability less than the minimum ability of the two. Geometrically, assume that $c(s_1) > c(s_2)$. Then the points on the graph of the membership function $c(s)$ must lie above the straight line $y = c(s_2)$ for all $s \in [s_1, s_2]$. Thus the set C_f with a membership function $c(s)$ as in FIGURE 5 is not a convex fuzzy set because $s_1 < s^* < s_2$ and $c(s^*) < c(s_2)$. Notice, however, that had $c(s)$ been convex and monotonic, i.e., without a minimum in (a, b), then C_f is a convex fuzzy set.

What if a set B_f were crisp? Then the membership function $b(s) = 1$ for all $s \in B$, and condition (1) holds. The region A equals $B \times [0, 1]$, therefore, A is convex. Condition (2) holds too; therefore B_f is convex. What can be said about fuzzy set operations on convex fuzzy sets? The answers are available in the literature [**2**], [**3**], [**7**], [**8**], [**9**]. But why not try your hand?

Concluding remarks

The simplicity of this informal introduction is appealing, yet it may not justify the tremendous growth of the subject in the past twenty years. Recently, fuzziness has received a lot of attention by the popular press [**12**], [**15**], [**16**], [**24**].

What does fuzzy set theory achieve that cannot be realized by other approaches? One strong reason for its importance is the fact that there is much need in modern applications to handle imprecision. In other words, the need exists for a systematic way to deal with ambiguity, to deal "with a wide variety of problem areas which do not lend themselves to precise analysis in the classical spirit" [**19**, Foreword by L. A. Zadeh]. For example, one needs the capability to work with fuzzy numbers when the situation is such that no definite value can be assigned to a quantity [**5**]. To clarify this point, assume that the value assigned to the measure of the probability for a certain event is 0.2351. How certain is this value? If there were a change in this value, how would the solution of the problem change? These are old questions with some answers in sensitivity analysis. Fuzzy set theory affords a successful new approach [**2**], [**8**].

What is there to read? Since 1965, some four thousand articles and thirty books have been published on the topic of fuzzy sets and related areas. Two international journals and an international society are entirely devoted to the support and the dissemination of the most recent advances in the field. A good way to begin, perhaps, is to consult a fairly complete and recent bibliography [7]. Although mostly concerned with a specialty topic, a good annotated bibliography is found in [19]. This text describes the implementation of a utility called the Fuzzy Risk Analyzer by the Computer Security Research Group at George Washington University. Other "expert systems" use fuzzy logic to troubleshoot diesel engines [4], or to analyze and diagnose certain diseases; see the papers [1], [6]. Since fuzzy logic provides the basic structure for a large majority of applications, good introductory reading consists of chapters III and IV of Kaufmann's text [9]. Algorithms for real-time control in automated manufacturing are obtained by using fuzzy digraphs, i.e., digraphs with a fuzzy vertex set and/or a fuzzy arc set [13], [14]. Semantic problems are handled by linguistic variables [23]. Interest in pattern recognition motivates the texts [2], [7]. On the more specific topic of digital image reconstruction, see the papers [17], [20]. And why not include some applications to mathematics too [10]?

It is a well-known fact that to write informally on a very familiar topic presents quite a challenge to any author. This author wishes to thank the referees, the editor, and Professor L. A. Zadeh.

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The Sure Thing?

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Lottery grand prizes are soaring; as of this writing the top grand prize has exceeded $40 million in Illinois. One may wonder: Does it pay to bet all possible combinations of numbers in order to win for sure? The cost of such a strategy is somewhat staggering, but if the winning ticket is the only winner and the grand prize is $15,000,000 or $20,000,000 then conventional wisdom suggests that the strategy may pay off handsomely. It is the purpose of this short paper to show that even if everything goes well and the system bettor is the only grand prize winner, the return on investment may not be as great as on other investments one may make. More than one winner may prove disastrous for the system bettor. The example which will be given involves the New York State Lotto, but the analysis could be used in just about any lottery in which it is possible to bet all possibilities.

In New York, a bettor in the Lotto picks any six integers between 1 and 44 inclusive. Note that the numbers cannot be repeated, and that the order of selection is irrelevant. The Lottery Commission randomly selects six of these forty-four integers as 'winning numbers' and a seventh as a 'supplementary number.' The player wins the grand prize if his selection matches all six winning numbers, second prize if five of the six winning numbers are matches, third prize if four of the six winning numbers are matches, and fourth prize if three of the six winning numbers and the supplementary number match.

In what follows, we will use the notation $C(n,r)$ to denote the number of different ways which r objects may be selected from n objects, regardless of the order of selection; thus

$$C(n,r)=\frac{n!}{r!(n-r)!}.$$

There are $C(44,6)=7{,}059{,}052$ ways of picking six different numbers out of forty-four. This is the total number of possibilities with which we deal. One of these selections will be a grand prize winner.

Since there are $C(6,5)=6$ ways of picking five winning numbers out of six and thirty-eight ways of picking one nonwinning number, there are, altogether, $38\times C(6,5)=228$ sets of numbers that will win second prize. Using similar reasoning, we obtain that there are $C(6,4)\times C(38,2)=10{,}545$ sets of numbers that will win third prize and $C(6,3)\times C(37,2)=13{,}320$ sets that will win fourth prize. (Note that in the last case there are $C(37,2)$ ways of picking the remaining nonwinning and nonsupplementary numbers.)

Summarizing, we can note: among the 7,059,052 possible choices,

1 wins first prize
228 win second prize
10,545 win third prize
13,320 win fourth prize.

In New York each game played costs $.50 (actually, $1.00 for the minimum of two games). Thus, the investment needed to cover all 7,059,052 possibilities is $3,529,526.

What can the player expect for this 'investment'? According to Lottery Commission rules, 38% of the revenue from the sale of tickets (40% less the 2% prize-reserve fund) is allocated as the winning pool for the draw. Of this winning pool, 12.5% is allocated equally among second prize winners, 25% among third prize winners, 12.5% among fourth prize winners. The first prize winners split 50% of the winning pool for the draw plus any first place money carried forward from previous draws. Thus the amount of money allocated depends upon the amount of money bet. Prize money per winning ticket does not seem to vary radically from draw to draw for second, third, and fourth prizes. More money bet means more money to allocate among more winners. Second prizes in different drawings do not differ by more than a factor of 2 or 3. Different third and fourth prizes may vary even less than this. These average payoffs seem to be approximately $735 for second prize, $32 for third prize and $12 for fourth prize. The theoretical average payoffs for second, third and fourth prizes are, respectively, $735.32, $31.80, and $12.59. The size of first place awards does vary, with prizes going up to $20 million and more per draw on occasion. This is due mainly to the fact that first prize money not won on previous draws is carried forward.

If the player does bet all combinations, then, based upon the computations done above, we know that:

a. He will win one first prize.
b. He will win 228 second prizes or about $167,580.
c. He will win 10,545 third prizes or about $337,440.
d. He will win 13,320 fourth prizes or about $159,840.

Thus the player can expect to win about $665,000 plus first prize money for his $3,529,526 investment. Let's see how this works out under various first prize scenarios. Note that first prize winners in New York State receive 1/20 of their prize immediately and the rest in nineteen equal yearly installments. These payments may be decreased due to tax withholding.

In order to determine whether it would be more advantageous to invest the amount $B =$ $3,529,526 by conventional methods or by betting in the New York Lotto we may argue as follows. If we invest B at an interest rate i, compounded annually, then at the end of 19 years we will have

$$B(1+i)^{19}. \tag{1}$$

On the other hand, if we bet the amount B in the Lotto using the scheme mentioned above and the grand prize is P, then we win a total of $P + 665{,}000$ dollars. By the rules of the New York Lotto this total will be paid out as follows: $I = P/20 + 665{,}000$ is paid out immediately. The remaining amount is paid in 19 equal yearly installments of $E = P/20$ dollars. (If there is tax withholding then E might not equal $P/20$ but would equal the amount paid per year after taxes are withheld.)

Suppose we may invest these payments at the interest rate i of (1) and that the money is compounded annually. Then the total value of this prize money at the end of the 19th year will be

$$I(1+i)^{19} + E\big((1+i)^{18} + (1+i)^{17} + \cdots + (1+i) + 1\big). \tag{2}$$

If B is larger than $P + 665{,}000$, then the amount in (1) is always larger than that in (2) and we can say that it is more advantageous to invest the amount B by conventional means. On the other hand, if I is larger than B then the Lotto investment is better no matter what the interest rate is. In all other cases we desire to determine for which value of i the quantity in (1) is equal to the quantity in (2). This will give us what is called the internal rate of return (IRR) of the B dollar investment. The IRR is one of the most used measures of return on investment. Thus we equate (1) and (2), use the identity

$$\frac{1-x^{19}}{1-x} = 1 + x + x^2 + \cdots + x^{18}$$

for $x = 1 + i$ (we assume $i \neq 0$), and obtain, after some algebraic manipulations,

$$(E - i(B - I))(1+i)^{19} - E = 0. \tag{3}$$

We shall apply equation (3) to a number of different scenarios and solve it in each instance by an iteration method. We used Newton's method programmed in Microsoft BASIC on an IBM personal computer to get the resulting returns.

Let's consider a fairly good case for the player. Assume that the first prize money is \$10,000,000 and that the player is the sole winner. We first look at this case when no income taxes are involved.

In this case the player bets $B = \$3{,}529{,}526$ and receives back $P + 665{,}000 = \$10{,}665{,}000$. This seems very good. However, he receives only $I = \$1{,}165{,}000$ back immediately. The rest comes in 19 equal annual installments of $E = \$500{,}000$. When we solve equation (3) with these values of B, E and I we see that i, the IRR of this flow of money, is approximately 20.53%. This is good, but someone with \$3.5 million to spend might do better running a business.

If two people split the \$10,000,000 then our player only receives \$5,665,000 back with \$915,000 coming immediately and the rest in 19 equal annual installments of \$250,000 each. If we solve equation (3) with B as above, with $I = \$915{,}000$ and $E = \$250{,}000$, this translates to a return of only 6.84%. He might as well invest his money in certificates of deposit or in an ordinary savings account.

The actual case is much worse than the above indicates, since the government wants its taxes. The first year's money should not be taxed as it may be offset by the huge bet. However, gambling losses cannot be carried forward and thus all winnings after the first year are taxed. Let's say the tax rate is 40%. Then in the first case the 19 payments are, in effect, only \$300,000 instead of \$500,000 each. We set E equal to \$300,000 and solve equation (3). This gives a rate of return of about 10.92%, which is not too good considering the fact that the player was the sole winner of \$10,000,000. In case there are two first prize winners, then, assuming a 40% tax rate on 19 payments of \$250,000 each, we may set $E = \$150{,}000$ in the second case. When we solve we calculate a rate of return of only a little more than .87%.

We found these numbers quite surprising. It seems that it is quite difficult to earn a decent living playing the Lotto in New York.

Note: Specific New York State lottery rules were obtained from "Play Lotto 6/44 by mail" effective October 29, 1983.

NOT COMING SOON

Editor's note: As a service to our readers, we provide a list below of titles of articles that will not be coming soon (or ever) in this MAGAZINE.

The Unit Ball as an R.O.T.C. Dance

Epsilons without Deltas—a Sociological Survey

Can a Subset of a Calculus Class be Nowhere Dense?

Successful Prosecutions under the Distributive Law

The Deviations of Statisticians Are Not Standard

Anarchistic Tendencies of Free Groups

—RICHARD LAATSCH
Miami University, Ohio

PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

LOREN LARSON, Editor-elect
St. Olaf College

Proposals

To be considered for publication, solutions should be received by April 1, 1986.

1226. Let

$$P(\lambda)=\int_0^{\infty}\frac{dx}{e^x+\lambda x}.$$

Find the Taylor expansion about the origin, and evaluate $P(1)$ and $P(-1)$ to ten decimal places. [*L. Matthew Christophe, Jr., Wilmington, Delaware.*]

1227. For real x let

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n}{1+n^2x^{2n}}.$$

Where is f continuous? [*G. A. Edgar, The Ohio State University.*]

1228. Let P be a stochastic matrix (all entries are nonnegative and each row sum is 1). Prove that for each positive integer n the largest entry in each column of $I+P+P^2+\cdots+P^n$ occurs on the diagonal. [*David Callan, University of Bridgeport.*]

1229. Let $P(x)$ be a polynomial of degree $n>0$ with coefficients in Q, the field of rational numbers. Let α be any complex number. Then $P(x)=(x-\alpha)q(x)+r$, where $q(x)=c_{n-1}x^{n-1}+c_{n-2}x^{n-2}+\cdots+c_1x+c_0$ and r is a complex number. Prove that the set $S=\{c_{n-1},c_{n-2},\ldots,c_0,r\}$ is linearly dependent over Q if and only if α is a zero of a polynomial of degree n in $Q[x]$. [*Roger L. Creech, East Carolina University.*]

1230. Let ABC and $A'B'C'$ be two similar and similarly oriented triangles in a plane. Let $AA'A''$, $BB'B''$, and $CC'C''$ be three triangles lying in the plane and similar and similarly oriented to ABC. Prove that triangle $A''B''C''$ is similar and similarly oriented to ABC. [*L. Kuipers, Sierre, Switzerland.*]

ASSISTANT EDITORS: DANIEL B. SHAPIRO and WILLIAM A. MCWORTER, JR., *The Ohio State University.*

*We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that the proposer did not supply a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

New proposals and solutions to problems numbered 1221 and higher should be sent to Loren Larson, Mathematics Dept., St. Olaf College, Northfield, MN 55057. Solutions to proposals numbered 1220 and lower should be sent to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210.

Quickies

Solutions to the Quickies are on p. 304

Q701. Show that among any eight composite integers selected from the first 360 natural numbers, there will always be two which are not relatively prime. [*Norman Schaumberger, Bronx Community College.*]

Q702. Are there any integral solutions to the Diophantine equation $x^2 + y^2 + z^2 = xyz - 1$? [*M. S. Klamkin, University of Alberta.*]

Q703. The equation $\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = 0$ is important in function theory (e.g., in some proofs of Jensen's formula). It is usually proved via an appeal to Cauchy's Integral Theorem. Give a short, "elementary" proof. [*Lawrence J. Wallen, University of Hawaii.*]

Solutions

A Hamiltonian Circuit — November 1984

1200. Let K_n be the complete graph on n vertices, and G any subgraph of K_n having at most $n-3$ edges. Then K_n has a Hamiltonian circuit not containing any edge of G. [*Paul Erdös, Hungarian Academy of Sciences.*]

Solution I: We proceed by induction on n and fix $n \geqq 7$, the cases $n \leqq 6$ being easily checked. Without loss of generality we can assume that G has exactly $n-3$ edges. Let g be the number of nonisolated vertices of G. Since $n-3 \geqq 4$, we have $g \geqq 4$. (If $g \leqq 3$, then G has at most three edges.) The average degree in G of nonisolated vertices of G is thus $2(n-3)/g \leqq (n-3)/2 < (n-1)/2$. Let v be a nonisolated vertex of G whose degree in G is less than $(n-1)/2$. The degree of v in $K_n - G$ is thus greater than $(n-1)/2$, since every vertex in K_n has degree $n-1$. Now apply the inductive hypothesis to $K_n - \{v\}$ with subgraph $G - \{v\}$, and obtain a Hamiltonian circuit H in $(K_n - G) - \{v\}$. Since v is adjacent to more than half the vertices of $K_n - \{v\}$, it must be adjacent to two consecutive vertices in H and thus can be spliced in to form a Hamiltonian circuit of K_n.

JERROLD W. GROSSMAN
Oakland University

Solution II: Let H be the complement of G in K_n. If H has no Hamiltonian circuit, then there are nonadjacent vertices u and v of H such that the sum of the degree of u and the degree of v is at most $n-1$ (Ore's theorem; see Behzad, Chartrand, and Lesniak-Foster, *Graphs and Digraphs*, pp. 137–138). In K_n, the sum of these degrees is $2n-2$, which counts the edge from u to v twice. Hence there are at least $(2n-2)-(n-1)-1 = n-2$ edges missing from H. But G has at most $n-3$ edges, a contradiction.

The problem is equivalent to exercise 7.8 on p. 145 of Behzad *et al.*

WILLIAM STATON
University of Mississippi

Also solved by Robert E. Bernstein, Stephen D. Bronn, Alberto Facchini (Italy), Ralph P. Grimaldi, Chico Problem Group, Dennis Hamlin (student), J. Metzger, Tom Mickiewich, Mike Molloy (student, Canada), Richard Parris, Robert Patenaude, Harry Sedinger, and the proposer. There were two incorrect or seriously incomplete solutions.

Parris noted that the number of deleted edges is as large as possible, since deleting $n-2$ edges adjacent to a single vertex of K_n would leave a graph without any Hamiltonian circuit. *The proposer* noted that stronger questions have been answered, such as the following. Let $G_1, \ldots, G_r$ be r edge-disjoint subgraphs of K_n such that every Hamiltonian circuit meets each of the G_i. If $e(G_i)$ denotes the number of edges of G_i, what is the minimum value of $\sum_{i=1}^{n} e(G_i)$?

A Tangent Sum **November 1984**

1201. Let

$$S_n = \sum_{k=0}^{n} 2^k \tan\frac{x}{2^k} \tan^2\frac{x}{2^{k+1}}.$$

Find simple expressions for S_n and $\lim_{n\to\infty} S_n$. [*Themistocles M. Rassias, Athens, Greece.*]

Solution: Since

$$\tan(2\alpha)\tan^2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\tan^2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} - 2\tan\alpha = \tan(2\alpha) - 2\tan\alpha$$

whenever α is not a nonzero integral multiple of $\pi/4$, we have

$$S_n = \sum_{k=0}^{n}\left(2^k\tan\frac{x}{2^k} - 2^{k+1}\tan\frac{x}{2^{k+1}}\right) = \tan x - 2^{n+1}\tan\frac{x}{2^{n+1}}$$

(a collapsing sum) whenever x is not a nonzero integral multiple of $\pi/2$. Since l'Hôpital's rule yields

$$\lim_{t\to 0}\frac{\tan(tx)}{t} = \lim_{t\to 0}\frac{x\sec^2(tx)}{1} = x$$

for all x, we have

$$\lim_{n\to\infty} S_n = \tan x - x$$

whenever x is not a nonzero integral multiple of $\pi/2$.

EDITOR'S COMPOSITE

Solved by Mangho Ahuja, Ricardo Alfaro, Frank P. Battles, Adanır Bilyer (Turkey), J. C. Binz (Switzerland), Stephen Wayne Coffman (student), Sheldon L. Degenhardt (student), Ralph Garfield, Steven J. Gustafson (student) & M. B. Gregory, Douglas Henkin (student), Víctor Hernández (Spain), Mark Kantrowitz (student), Benjamin G. Klein, Kenneth A. Klinger, L. Kuipers (Switzerland), J. C. Linders (The Netherlands), James Magliano, Hosam M. Mahmoud, Syrous Marivani, Vania D. Mascioni (student, Switzerland), Roger B. Nelsen, William A. Newcomb, Paul O'Hara, Richard Parris, P. J. Pedler (Australia), Bjorn Poonen (student), Harvey Schmidt, Jr., Sahib Singh, John S. Sumner, John Tolppi (student), Douglas H. Underwood, Michael Vowe (Switzerland), Gary L. Walls, Yan-Loi Wong (student), Jihad Yamout (student), Robert L. Young, and the proposer.

All solvers used essentially the same method, but only a few stated conditions under which the manipulations are valid. Many solvers used the fact that $\lim_{t\to 0}((\tan t)/t) = 1$ with $t = x/2^{n+1}$ without noticing that it is invalid when $x = 0$. *Mascioni & H. Kappus* (Switzerland) found the formula for S_n in E. R. Hansen, *A Table of Series and Products*, p. 269, formula 37.2.3.

A Cosine Algorithm **November 1984**

1202. Let m and n be coprime positive integers with $n > m$. Let θ be a real number. Consider the following algorithm:

(i) Initialize (A, B, C, d, e) to $(2\cos\theta, 2\cos\theta, 2, n-m, m)$.
(ii) If $d > e$, replace (B, C, d) by $(AB - C, B, d - e)$; otherwise replace (A, C, e) by $(AB - C, A, e - d)$.
(iii) If $e = 0$, terminate the algorithm; otherwise return to step (ii).

Prove that $A = 2\cos(n\theta)$ when the algorithm terminates. [*Peter L. Montgomery, System Development Corporation, Santa Monica, California.*]

Solution: Define three new variables α, β, γ, initialized to 1,1,0. In step (ii) of the algorithm, replace (β, γ) by $(\alpha+\beta, \beta)$ if $d > e$; otherwise replace (α, γ) by $(\alpha+\beta, \alpha)$. By an easy induction, the following relations are valid at each step:

$$\alpha d + \beta e = n,$$
$$\gamma = |\alpha - \beta|,$$
$$A = 2\cos(\alpha\theta), \qquad B = 2\cos(\beta\theta), \qquad C = 2\cos(\gamma\theta).$$

(The first comes from $\alpha(d-e)+(\alpha+\beta)e = \alpha d + \beta e = (\alpha+\beta)d + \beta(e-d)$ and the third comes from the identity $2\cos(\alpha\theta)\cos(\beta\theta) - \cos((\alpha-\beta)\theta) = \cos((\alpha+\beta)\theta)$.) The part of the algorithm dealing with d and e is simply the standard Euclidean algorithm. Therefore, when termination occurs, we have $d = \gcd(m, n) = 1$, $e = 0$, $\alpha = n$, and $A = 2\cos(n\theta)$.

WILLIAM A. NEWCOMB
Lawrence Livermore National Laboratory

Also solved by J. C. Binz (Switzerland), Stephen D. Bronn, J. C. Linders (The Netherlands), Richard Parris, Bjorn Poonen (student), and the proposer. There was one incorrect solution.

Parris found the editor's error: step (ii) is used only if $e > 0$. *Bronn* provided a generalization: if (A, B, C, d, e) is initialized to $(2\cos a, 2\cos b, 2\cos(b-a), N, M)$, where N and M are relatively prime positive integers, and the algorithm is otherwise unchanged, then the value of A at termination is $2\cos(Na + Mb)$. *Parris* and *the proposer* noted the relation of the algorithm to Chebyshev polynomials. The proposer also found additional invariant relations: $m\alpha \equiv e \pmod n$ and $m\beta \equiv -d \pmod n$. The algorithm was discovered while he was investigating efficient ways to compute values of the Lucas functions, $V_n(P) \equiv \alpha^n + \alpha^{-n} \pmod N$, where $\alpha + \alpha^{-1} = P$, which are used in the $p+1$ method of factoring.

Quadratic Polynomial Iteration — November 1984

1203. Let $p(x) = ax^2 + bx + c$, where a, b, and c are integers with $a \neq 0$. If n is an integer such that $n < p(n) < p(p(n))$, show that $p(p(n)) < p(p(p(n)))$ if and only if $a > 0$. [*B. Landman and J. Layman, Virginia Polytechnic Institute and State University, and B. Klein, Davidson College.*]

Solution: We abbreviate $p(p(x))$ as $p^2(x)$ and $p(p(p(x)))$ as $p^3(x)$. We note first that if $x \neq y$, then

$$\frac{p(x)-p(y)}{x-y} = a(x+y)+b.$$

Since the integer a is nonzero, it is sufficient to show that $p^2(n) < p^3(n)$ if and only if $a > -1$. Now

$$\frac{p^3(n)-p^2(n)}{p^2(n)-p(n)} - \frac{p^2(n)-p(n)}{p(n)-n} = a(p^2(n)+p(n)) + b - (a(p(n)+n)+b) = a(p^2(n)-n).$$

Then

$$p^3(n)-p^2(n) = (p^2(n)-p(n))\left(a(p^2(n)-n) + \frac{p^2(n)-p(n)}{p(n)-n}\right).$$

Thus,

$$p^3(n) > p^2(n) \Leftrightarrow a(p^2(n)-n) + \frac{p^2(n)-p(n)}{p(n)-n} > 0$$
$$\Leftrightarrow a > \frac{-(p^2(n)-p(n))}{(p^2(n)-n)(p(n)-n)}.$$

But

$$0<\frac{p^2(n)-p(n)}{(p^2(n)-n)(p(n)-n)}<\frac{1}{p(n)-n}\leqq 1.$$

Hence $p^3(n)>p^2(n)$ if and only if $a>-1$, since a, n, $p(n)$, and $p^2(n)$ are integers.

HISASHI YOKOTA
South Dakota State University

Also solved by David Callan, Mark Kantrowitz (student, $a>0$ only), J. C. Linders (The Netherlands), Syrous Marivani, Richard Parris, Bjorn Poonen (student), Harvey Schmidt, Jr., Dennis Spellman, and the proposer. There were three incorrect or seriously incomplete solutions.

The editor misread and thereby weakened the problem. The proposers' conclusion was: if $a>0$, then $p^3(n)>p^2(n)$; but if $a<0$, then $p^3(n)\leqq p(n)<p^2(n)$.

Euler's Constant, Expanded — November 1984

1204. Euler's constant γ is defined by

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right). \tag{1}$$

Prove the following generalization of (1):

$$\gamma=\lim_{n\to\infty}\left[\left(\frac{2^m}{1}+\frac{3^m}{2}+\cdots+\frac{n^m}{n-1}\right)+m-\left(S_n^0+S_n^1+\cdots+S_n^{m-1}\right)-\ln(n-1)\right],$$

where $S_n^m=1^m+2^m+\cdots+n^m$. [*H. Roelants, Hoger Instituut voor Wijsbegeerte, Leuven, Belgium.*]

Solution: Since

$$\sum_{k=0}^{m-1}S_n^k=\sum_{k=0}^{m-1}\sum_{j=1}^{n}j^k=\sum_{j=1}^{n}\sum_{k=0}^{m-1}j^k=\sum_{k=0}^{m-1}1^k+\sum_{j=2}^{n}\sum_{k=0}^{m-1}j^k$$

$$=m+\sum_{j=2}^{n}\frac{j^m-1}{j-1}=m+\sum_{j=2}^{n}\frac{j^m}{j-1}-\sum_{j=2}^{n}\frac{1}{j-1},$$

we obtain

$$\lim_{n\to\infty}\left(\sum_{j=2}^{n}\frac{j^m}{j-1}+m-\sum_{k=0}^{m-1}S_n^k-\ln(n-1)\right)$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\ln(n-1)\right)=\gamma.$$

MICHAEL VOWE
Therwil, Switzerland

Also solved by Adanır Bilyer (Turkey), Stephen Wayne Coffman (student), Steven Davi, Thomas P. Dence, Riad Ghibril (student, Lebanon), Chico Problem Group, San Bernardino Problem Solving Group, Douglas Henkin (student), Víctor Hernández (Spain), Mark Kantrowitz (student), L. Kuipers (Switzerland), J. C. Linders (The Netherlands), Syrous Marivani, Vania D. Mascioni (student, Switzerland), Jack McCown, Mike Molloy (student, Canada), William A. Newcomb, Paul O'Hara, Richard Parris, P. J. Pedler (Australia), Stephen Penrice (student), Bjorn Poonen (student), Vincent P. Schielack, Jr., Jan Söderkvist (student, Sweden), and the proposer.

1205. Show that for $x \in [0,1]$

$$\text{(a)} \qquad \left|\frac{\pi}{4} - \operatorname{Arcsin} x\right| \leqq \frac{\pi}{4}\sqrt{1 - 2x\sqrt{1-x^2}} \quad \text{and}$$

$$\text{(b)} \qquad \operatorname{Arcsin} x \leqq \frac{x}{\dfrac{2}{\pi} + \dfrac{\pi}{12}(1-x^2)}.$$

When does equality hold? [*Vania D. Mascioni, student, ETH Zürich, Switzerland.*]

Solution I: (a) Let $x = \sin\varphi$, where $0 \leqq \varphi \leqq \pi/2$, and let the function f be defined by $f(0) = 1$ and $f(t) = (\sin t)/t$ for $t \neq 0$. Since

$$\left|\frac{\pi}{4} - \operatorname{Arcsin} x\right| = \left|\frac{\pi}{4} - \varphi\right|$$

and

$$\sqrt{1 - 2x\sqrt{1-x^2}} = \sqrt{1 - 2(\sin\varphi)(\cos\varphi)} = |\cos\varphi - \sin\varphi| = \sqrt{2}\left|\sin\left(\frac{\pi}{4} - \varphi\right)\right|,$$

the inequality becomes

$$\sin t \geqq \frac{4}{\pi\sqrt{2}}t \quad \text{for } 0 \leqq t = \left|\frac{\pi}{4} - \varphi\right| \leqq \frac{\pi}{4}.$$

Since f is decreasing on $[0, \pi/4]$ and $f(\pi/4) = 2\sqrt{2}/\pi$, the inequality is clear. Equality occurs where $t = 0$, i.e. $x = 1/\sqrt{2}$ (maximum value of f), and where $t = \pi/4$, i.e., $x = 0$ or $x = 1$ (minimum value of f).

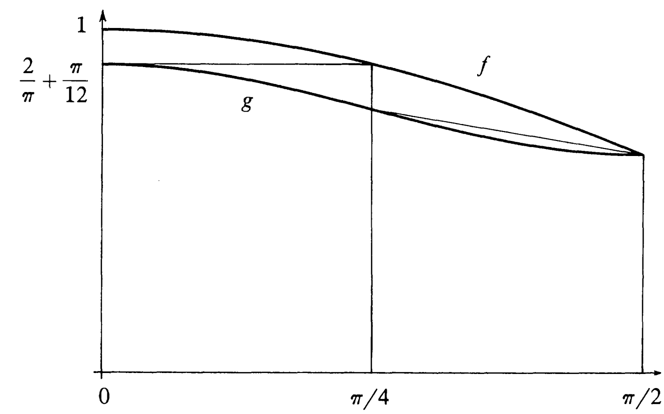
Robert E. Shafer
Berkeley, California

Solution II: (b) With $x = \sin\varphi$ and f as in Solution I, the inequality to be proved reduces to $f(\varphi) \geqq g(\varphi)$ for $0 < \varphi \leqq \pi/2$, where

$$g(\varphi) = \frac{2}{\pi} + \frac{\pi}{12}\cos^2\varphi.$$

(Equality obviously holds if $x = 0$.) Since $g'(\varphi) = -(\pi/12)\sin(2\varphi)$, g is decreasing on the interval $[0, \pi/2]$. If $\varphi \in [0, \pi/4]$, then

$$g(\varphi) \leqq g(0) = \frac{2}{\pi} + \frac{\pi}{12} < .9 < \frac{4}{\pi\sqrt{2}} = f\left(\frac{\pi}{4}\right) \leqq f(\varphi).$$



Since $g''(\varphi) = -\frac{1}{6}\pi\cos(2\varphi)$, we have $g''(\varphi) > 0 > f''(\varphi)$ for $\varphi \in (\pi/4, \pi/2)$. Since $g(\pi/2) = 2/\pi = f(\pi/2)$ and $g(\pi/4) < f(\pi/4)$, we see that the graph of g on $(\pi/4, \pi/2)$ is below the line joining $(\pi/4, g(\pi/4))$ with $(\pi/2, 2/\pi)$, whereas the graph of f is above it. Hence $f(\varphi) > g(\varphi)$ for $\varphi \in (\pi/4, \pi/2)$. We have $f(\varphi) = g(\varphi)$ if and only if $\varphi = \pi/2$, as is seen from the graph. Hence the inequality to be proved becomes an equality if and only if $x = 0$ or $x = 1$.

L. KUIPERS
Sierre, Switzerland

Also solved by Erhard Braune (Austria), Chico Problem Group, Hans Kappus (Switzerland, part (a) only), Syrous Marivani, William A. Newcomb, Bjorn Poonen (student), Jan Söderkvist (student, Sweden), Michael Vowe (Switzerland), and the proposer. Late solution by David Paget (Australia, part (b) only).

Braune and the proposer note that (b) follows from *Monthly* problem E2952, proposed v. 89 (1982), p. 424 (solution not yet published). Vowe derived (a) from Aufgabe 25, §24, p. 104 of A. Ostrowski, *Aufgabensammlung zur Infinitesimalrechnung I*, and (b) from inequality 3.4.15, p. 238 of D. S. Mitrinović, *Analytic Inequalities*. Newcomb showed that inequality (b) holds (with the same equality conditions) when $\pi/12$ is replaced by any number between $1/6$ and $1-2/\pi$; for Poonen the replacement bounds were $\pi/12$ and $1-2/\pi$; and for Vowe, 0 and $1/\pi$.

Answers

Solutions to the Quickies on p. 299.

Q701. Since $19^2 = 361$, any composite less than 361 is divisible by some prime less than 19. Since there are precisely seven primes less than 19, at least two of the eight composites must be divisible by the same prime.

Q702. By parity considerations, exactly two of x, y, z must be even. Hence by the substitutions $x = 2a, y = 2b, z = 2c+1$, the equation reduces to $4a^2 + 4b^2 + 4c^2 + 4c + 2 = 4ab(2c+1)$. This is impossible.

Q703. Absolute convergence of the integral follows from $|e^{i\theta} - 1| \sim |\theta|$ as $\theta \to 0$. Rotate by π to obtain

$$\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = \int_0^{2\pi} \log|1 + e^{i\theta}|\, d\theta = c.$$

Now add to obtain

$$2c = \int_0^{2\pi} \log(|1 + e^{i\theta}|\,|1 - e^{i\theta}|)\, d\theta = \int_0^{2\pi} \log|1 - e^{2i\theta}|\, d\theta = \frac{1}{2}\int_0^{4\pi} \log|1 - e^{i\psi}|\, d\psi = c,$$

by periodicity. Hence $c = 0$.

Note: From this it follows immediately that

$$\int_0^{\pi} \log\sin\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \log\sin\frac{\theta}{2}\, d\theta = \frac{1}{2}\int_0^{2\pi} (\log|1 - e^{i\theta}| - \log 2)\, d\theta = -\pi\log 2.$$

Ed. note. A similar proof can be found in *Lectures on Functions of a Complex Variable*, Gerretsen and Sansone, v. 1, pp. 151–152.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Cartwright, Mark, *Ten thousand pages to prove simplicity*, New Scientist (30 May 1985) 26-30.

Popular but serious account of the history of classification of simple groups, illustrated with photographs of early contributors. (Sadly, this is the last in a series of articles New Scientist has been publishing on mathematics and its applications. The 9 articles have been collected and reprinted in a booklet, available from Keith Jones, New Science Publications, Commonwealth House, 1-19 New Oxford St., London WC1 1NG, for £2 (£2.50 outside U.K.), with cheque payable to "New Scientist.")

Dewdney, A. K., *Computer recreations: analog gadgets ...*, Scientific American 252:6 (June 1985) 18-29, 136.

In today's world of digital this and that, do analog devices still have a place? Dewdney describes analog gadgets sent in by readers, before proceeding to a short discussion of the heart of the question. Along the way, an alien named Martian [sic] Gardner makes a cameo appearance; at the conclusion, NP-completeness lies besieged by an analog attack whose success still hangs in the balance.

Dewdney, A. K., *Computer recreations: A computer microscope zooms in for a look at the most complex object in mathematics*, Scientific American 253:2 (August 1985) 16-24, 120.

The "most complex object in mathematics"? Why, the Mandelbrot set, of course. Dewdney tells how to bring up pictures of this amazing set on your personal computer, and gives an address where professional-quality portraits of it can be obtained.

Anderson, John R., et al., *Intelligent tutoring systems*, Science 228 (26 April 1985) 456-462.

Describes successes with two computer tutoring systems for teaching geometry proofs and Lisp programming. Each system uses several hundred forward and backward inference rules, ordered according to "aptness."

Kolata, Gina, *Number theory connections*, Science 228 (17 May 1985) 833-834.

Account of an unusually-structured meeting in honor of John Tate, the meeting's theme being the connections between number theory and diverse fields of mathematics.

Kolata, Gina, *Making factoring easier*, Science 228 (19 April 1985) 310.

H. Lenstra has devised yet another technique for fast factoring; this one uses elliptic curves. The method is really relevant only to "pure" factoring, as it is very fast only for numbers whose prime factors are of different sizes. For numbers with nearly same-size prime factors--the domain of the "applied" factoring for cryptographic purposes--Lenstra's method is "only about as fast as the best existing methods." Says Carl Pomerance (Georgia), however: "The big breakthrough in factoring hasn't come yet... Maybe 10 years from now, people won't be talking about factoring because it will be easy to do."

Kolata, Gina, *Must "hard problems" be hard?*, Science 228 (26 April 1985) 479-481.

Does P equal NP? Or is there a hierarchy of distinct classes of difficulty of problems? A. Yao (Stanford) has shown that there is a hierarchy for the restricted universe of problems which admit "oracles." Interestingly enough, there is a small practical application of this highly abstract research: "Everyone knew that you can't do parity or multiplication on a programmable logic array, but no one could prove it. Now we know why these things are impossible," says Michael Sipser (MIT).

Kolata, Gina, *Solving linear systems faster*, Science 228 (14 June 1985) 1297-1298.

Both Gaussian elimination and iterative methods have drawbacks as solution techniques for linear systems. V. Pan (SUNY-Albany) and J. Reif (Harvard) have discovered another, faster approach, which relies on a large number (n^3) of parallel processors.

Ennis, John, *Statistics, St. Petersburg and Sellafield*, New Scientist (2 May 1985) 26-28.

"Most results in the theory of probability and statistics seem to run completely counter to common sense. Because politicians do not understand the subject, they are wasting public money and human lives."

La Brecque, Mort, *Fractal symmetry*, Mosaic 16:1 (January/February 1985) 14-23.

Mandelbrot's fractals are finding their way into the mainstream of physics of critical phenomena, including percolation clusters, polymers, and irreversible kinetic processes. Fractals may also provide the unifying concept behind all amorphous materials (such as glasses). Raymond Orbach guesses that "fractals are very general indeed, occurring more often in nature even than do the regular atomic and molecular arrays of traditional physics." (Francophone readers may also enjoy the French-language cartoon that introduces children to fractals on pp. 10-13 preceding this article.)

Sander, L. M., *Fractal growth processes, Physics News in 1984*, Physics Today (January 1985) 5-19.

A computer simulation of a growth model called DLA (for "diffusion-limited aggregation") suggests that a lightning bolt is probably a fractal of dimension 2.4. Other growth processes, such as for clustering of smoke particles, also appear to be fractals.

Robinson, Arthur L., *Fractal fingers in viscous fluids*, Science 228 (31 May 1985) 1077-1080.

Immisicible fluids interpenetrate each other in an array of "fingers," which appear to have the self-similarity of fractals.

Tape, Walter, *The topology of mirages*, Scientific American 252:6 (June 1985) 120-123, 127-129, 136.

Splendid article showing geometrically how the folds of transfer mappings produce the qualitative phenomena we can observe in mirages. Featured are illustrative photographs by the author and a 1981 "odd-number theorem" proved by William L. Burke, an astronomer. The theorem says that a smooth transfer mapping produces an odd number of images, and the reader can count them in the photographs.

Fisher, Arthur, *Chaos: the ultimate asymmetry*, Mosaic 16:1 (January/February 1985) 24-33.

Joseph Ford (Georgia Tech) "believes, as do others, that chaos theory as it is now developing represents a third revolution in physics. In this regard it is similar in importance to relativity and quantum mechanics, and it has the same probability of forever changing our picture of the universe." But it is not just physics that has a stake in the mathematics of chaos: "If we have chaos in the equations," says Doyne Farmer, "...then what happens in the future is critically dependent on very small choices that we make... Each of us has had the experience of something trivial being a turning point in our lives..."

Holden, Arun, *Chaos is no longer a dirty word*, New Scientist (25 April 1985) 12-17.

Describes mathematical chaos in terms a math student will appreciate (i.e., algebra is not avoided) and notes the significance of chaos for science (e.g., a well-controlled experiment need not be reproducible).

Simons, Geoff, The Biology of Computer Life: Survival, Emotion, and Free Will, Birkhäuser, 1985; xii + 236 pp, $16.95.

Simons, the author of Are Computers Alive? begins "The doctrines of computer life are not congenial to many people." They are often already anxious or suspicious about existing machines being "more intelligent" than themselves; they often take refuge in a sense of human uniqueness derived from the idea that they have emotions but machines can't. Simons does his best to kick away this crutch, as he traces the evolution of "artificial emotion," as well as the survival strategies, growing autonomy, and developing creativity of computers. So, this book is not for the computerphobic or the timid of imagination. And us mathematicians? As heirs of "question everything!" Descartes, we are the most bound to show tolerance for such an advocate of the devil--er machine.

Gardner, Martin, The Magic Numbers of Dr. Matrix, Prometheus, 1985; 326 pp, $19.95, $10.95 (P).

There have been two earlier collections--The Numerology of Dr. Matrix (1967) and The Incredible Dr. Matrix (1976)--but this third one is the only complete account of Martin Gardner's friendship with Dr. Irving Joshua Matrix, the famous numerologist, over the last 20 years of the latter's life.

5x5x5 Cube (available from C. Bandelow, Haarholzer Str. 13, 4630 Bochum-Stiepel, W. Germany); $15, plus postage.

Invented by U. Krell in Hamburg, and produced by U. Meffert in Hong Kong, this puzzle has 87 moving parts. Only 2000 of these cubes were made, and a few hundred remain. If you want one, get one while they last.

Siamese Cubes (available from C. Bandelow, Haarholzer Str. 13, 4630 Bochum-Stiepel, W. Germany); $12 per joined pair, plus postage.

Rubikmania came and went; when did you last see a puzzle in a store? Here are two Rubik-like cubes joined by having the three cubies of one edge in common.

NEWS & LETTERS

A NEW PENTAGON TILER

The pentagon tiling shown on our cover was discovered this year by Rolf Stein, of the University of Dortmund. The pentagon is not one of the 13 types previously known to tile the plane (see [1]), and is the unique pentagon which satisfies these relations on sides and angles:

$$A = \pi/2,\ C + E = \pi,\ 2B + C = 2\pi$$
$$d = e = 2a = a + c.$$

[1] D. Schattschneider, "Tiling the Plane with Congruent Pentagons", this MAGAZINE, 51 (1978) 29-44.

RECREATIONAL MATHEMATICS SOURCES

I am engaged in a project to find the sources of classical problems in recreational mathematics, first compiling an annotated bibliography of the material. My computer file has some 160 subjects; I would be delighted to hear from anyone interested in this project.

David Singmaster
Polytechnic of the South Bank
London SE1 0AA England

SMOOTHING A SQUARE

Theorem 1 of "Differentiability and the Arc Chord Ratio" (this MAGAZINE, MAY 1985, 166-170) should have the added hypothesis that C has nonzero tangent everywhere. The need for the hypothesis is shown by noting that the square can be parametrized smoothly (take $x(t) = \max(t^3,0)$, $y(t) = \min(t^3,0)$ for $|t| < \eta$ and similar formulas for the other corners).

Joel Zeitlin

OLYMPIAD NEWS

The November 1985 College Mathematics Journal focuses on the 1985 IMO, featuring an interview with the USA team (2nd place victors), profiles of previous Olympiad winners, and articles on problem solving. Here are the IMO problems; solutions to USA and Canadian Olympiads follow.

26th INTERNATIONAL MATH OLYMPIAD HELSINKI, FINLAND, JULY 1985

1. A circle has centre on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

2. Let n and k be given relatively prime natural numbers, $0 < k < n$. Each number in the set $M = \{1, 2, \ldots, n-1\}$ is coloured either blue or white. It is given that
   (i) for each $i \in M$, both i and $n-i$ have the same colour, and
   (ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.

   Prove that all numbers in M must have the same colour.

3. For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, 2, \ldots$ let $Q_i(x) = (1 + x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers such that $0 \leq i_1 < i_2 < \ldots < i_n$, then
$$w(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq w(Q_{i_1}).$$

4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

5. A circle with centre O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

6. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n \cdot (x_n + \frac{1}{n})$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.

1985 USA AND CANADIAN OLYMPIADS: SOLUTIONS

The solutions which follow have been especially prepared for publication in the MAGAZINE by Loren Larson and Bruce Hanson of St. Olaf College.

14th USA MATH OLYMPIAD

1. Determine whether or not there are any positive integral solutions of the simultaneous Diophantine equations

$$x_1^2 + x_2^2 + \ldots + x_{1985}^2 = y^3,$$

$$x_1^3 + x_2^3 + \ldots + x_{1985}^3 = z^2,$$

such that $x_i \neq x_j$ for all $i \neq j$.

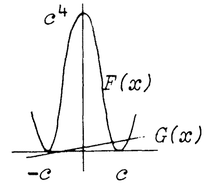
Sol. The second equation can be satisfied by taking $x_i = i$. This suggests that we consider $x_i = ci$ for a constant c yet to be determined. We have $\sum_{i=1}^{n} x_i^3 = c^3(n(n+1)/2)^2$, and $\sum_{i=1}^{n} x_i^2 = c^2K$ where $K = \sum_{i=1}^{n} i^2$. Both of these equations are satisfied by taking $c = K^4$ ($z = K^6 \frac{n(n+1)}{2}$ and $y = K^3$).

2. Determine each real root of $x^4 - (2\cdot 10^{10}+1)x^2 - x + 10^{20} + 10^{10} - 1 = 0$ correct to four decimal places.

Sol. Let $a = 10^5$. The equation can be written in the form

$$P(x) \equiv (x^2-a^2)^2 - (x^2-a^2) - x - 1$$
$$= (x^2-a^2-\tfrac{1}{2})^2 - x - \tfrac{5}{4} = 0.$$

The graphs of $F(x) = (x^2-c^2)^2$, $c^2 = a^2 + \frac{1}{2}$, and $G(x) = x + \frac{5}{4}$, make it clear that $P(x) = F(x) - G(x) = 0$ has exactly two real roots and these are very close to a.



So, let $r = a + e$, and suppose that $P(r) = 0$. Then $F(r) = G(r)$. We expect e to be small so that $G(r) = a + e - 5/4 \approx a$, and therefore $F(r) = ((a+e)^2 - e^2 - \frac{1}{2})^2 = (2ae+e^2-\frac{1}{2})^2 \approx 4a^2e^2$. Consequently $4a^2e^2 \approx a$, or equivalently, $e \approx \frac{\pm 1}{2\sqrt{a}}$.

Thus, we expect the roots to be close to $a \pm \frac{1}{2\sqrt{a}} = 10^5 \pm \frac{5\sqrt{10}}{10{,}000}$. Now $3.1 < \sqrt{10} < 3.3$, and therefore $\frac{15.5}{10{,}000} < \frac{5\sqrt{10}}{10{,}000} < \frac{16.5}{10{,}000}$. This leads us to investigate the values of P at the points $r_1 = 10^5 - 165\cdot 10^{-5}$, $r_2 = 10^5 - 155\cdot 10^{-5}$, $r_3 = 10^5 + 155\cdot 10^{-5}$, $r^4 = 10^5 + 165\cdot 10^{-5}$. It is routine to verify that $P(r_1) > 0$, $P(r_2) < 0$, $P(r_3) < 0$, $P(r_4) > 0$. Hence there is a root between r_1 and r_2, and it equals $10^5 - 16\cdot 10^{-4} = 99999.9984$, correct to four decimal places, and another root between r_3 and r_4, which equals $10^5 + 16\cdot 10^{-4} = 100000.0016$, correct to four decimal places.

3. Let A, B, C and D denote any four points in space such that at most one of the distances AB, AC, AD, BC, BD and CD is greater than 1. Determine the maximum value of the sum of the six distances.

Sol. Suppose the distance between A and D is allowed to be greater than 1. It is easy to see that if A, B, C are fixed, the maximum sum of distances will occur when D is in the plane of A, B, C (swing ΔBCD about axis BC). Now draw unit circles about points A and D. Points B and C must lie within the intersection of these circles, and the maximum sum of distances will occur when B and C are at the points of intersection of the two circles. We may now suppose that the distances AB, AC, BD, and CD are each equal to 1.

Let BC and AD intersect at O, and let $\theta = \angle BAO$. Then $AD + BC = 2(\sin\theta + \cos\theta) = 2\sqrt{2}\sin(\theta+45°)$. The restriction that $BC \leq 1$ translates into $0° \leq \theta \leq 30°$, and since sine is increasing between 0° and 90°, the maximum of $AD + BC$ occurs when $\theta = 30°$. Thus, the maximum of $AB + AC + BD + CD + AD + BC$ is $5 + \sqrt{3}$.

4. There are n people at a party. Prove that there are two people such that, of the remaining $n - 2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them, each of whom either knows both or else knows neither of the two. Assume that "knowing" is a symmetric relation, and that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Sol. Given two people at the party, describe a third party as *consistent* if that person knows either both or neither of the two people. A person who knows exactly k people at the party is consistent with respect to $\binom{k}{2} + \binom{n-1-k}{2}$ pairs. A calculation shows that $\binom{k}{2} + \binom{n-1-k}{2} = \frac{1}{2}[k^2 + (n-1-k)^2 - (n-1)] \geq$

$$\frac{1}{2}\left[2\left(\frac{n-1}{2}\right)^2 - (n-1)\right] = \frac{(n-1)(n-3)}{4}.$$

By the pigeonhole principle, there is a pair with which at least

$$\left\lceil \frac{n(n-1)(n-3)}{4\binom{n}{2}} \right\rceil = \left\lceil \frac{n-3}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ of the}$$

remaining people are consistent (where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

5. Let $a_1, a_2, a_3, \ldots$ be a non-decreasing sequence of positive integers. For $m \geq 1$, define $b_m = \min\{n: a_n \geq m\}$, that is, b_m is the minimum value of n such that $a_n \geq m$. If $a_{19} = 85$, determine the maximum value of $a_1+a_2+\ldots+a_{19}+b_1+b_2+\ldots+b_{85}$.

Sol. More generally, let q be an arbitrary positive integer and set $p = a_q$. We will show that for any non-decreasing sequence of positive integers with $a_q = p$, $a_1 + \ldots + a_q + b_1 + \ldots + b_p = p(q+1)$.

To do this, divide a p by q rectangle into pq congruent unit squares and color them white. Label the columns left to right from 1 to q, and the rows bottom to top from 1 to p. For each $n=1,\ldots,q$, blacken the squares from 1 to a_n in the nth column. The black squares form a nondecreasing staircase from left to right, and in the mth row, the first b_m-1 squares are white, and the last $q-(b_m-1)$ squares are black. The rectangle therefore has $a_1+a_2+\ldots+a_q$ black squares and $(b_1-1)+\ldots+(b_p-1)$ white squares. Thus $(a_1+\ldots+a_q) + [(b_1-1)+\ldots+(b_p-1)] = pq$, or equivalently, $(a_1+\ldots+a_q) + (b_1+\ldots+b_p) = p(q+1)$. In particular, if $a_{19} = 85$, we have $p(q+1) = 1700$.

17th CANADIAN MATH OLYMPIAD

1. The lengths of the sides of a triangle are 6, 8 and 10 units. Prove that there is exactly one straight line which simultaneously bisects the area and perimeter of the triangle.

Sol. Let the vertices of the triangle be A, B, C, with $AB = 8$ and $AC = 6$. If P is any point on the perimeter, we let Q denote the corresponding point on the perimeter for which the line PQ bisects the perimeter. It suffices to consider those points P which lie on the legs of the triangle within 6 units of A.

Suppose P is on AB and let $AP = x$, $0 \leq x \leq 6$. Then Q is on BC and $BQ = 4+x$. Then Area $\Delta PBQ = \frac{1}{2}(8-x)\frac{3}{5}(4+x) = \frac{3}{10}[36 - (x-2)^2] < 12 = \frac{1}{2}$ Area ΔABC.

Let D be on AC with $AD = 4$ and let E be on AB with $AE = 6$. Suppose P is between D and C. Then Q is between E and B and Area $\Delta PAQ >$ Area $\Delta DAE = 12$.

Suppose P is on AC and let $AP = y$, $0 \leq y \leq 4$. Then Q is on CB and $CQ = 6+y$. Then Area $\Delta PCQ = \frac{1}{2}(6-y)\frac{4}{5}(6+y) = \frac{2}{5}(36-y^2) = 12$ if and only if $y = \pm\sqrt{6}$.

The preceding shows that there is precisely one line which bisects the area and perimter of the triangle (it intersects AC $\sqrt{6}$ units from A).

2. Prove or disprove that there exists an integer which is doubled when the initial digit is transferred to the end.

Sol. Let N be an arbitrary positive integer with decimal representation $a_n a_{n-1} \ldots a_2 a_1 a_0$. We must decide if it is possible that

$$2(10^n a_n + \ldots + 10a_1 + a_0) =$$

$$10^n a_{n-1} + \ldots + 10^2 a_1 + 10a_0 + a_n. (*)$$

The left side of (*) is even, and therefore the units digit on the right side must be even. Thus $a_n = 2,4,6,$ or 8.

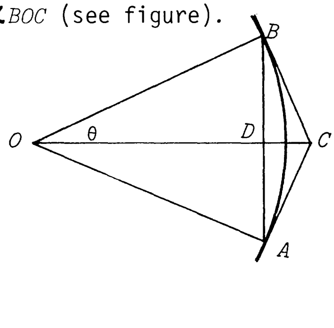
Suppose $a_n = 6$ or 8. Then $2N > 10^{n+1}$ and (*) cannot hold because the right side is less than 10^{n+1}.

Suppose $a_n = 2$. Then $a_0 = 1$ or 6. Taking (*) modulo 100 yields $20a_1 \equiv 8a_0 + 2 \pmod{100}$. This congruence cannot hold when $a_0 = 1$ or 6.

Suppose $a_n = 4$. Then $a_0 = 2$ or 7. Taking (*) modulo 1000 yields $200\,a_2 \equiv 80a_1 + 8a_0 + 4 \pmod{1000}$. This congruence cannot hold when $a_0 = 2$ or 7. We conclude that (*) can never be satisfied.

3. Let P_1 and P_2 be regular polygons of 1985 sides and perimeters x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and this circle passes through each vertex of P_2. Prove $x + y \geq 2c$. (You may assume that $\tan\theta \geq \theta$ for $0 \leq \theta < \pi/2$.)

Sol. Let O denote the center of the given circle, which we may assume without loss of generality has radius 1. Let A and B denote consecutive vertices of P_2, and position P_1 so that P_2 is tangent to the circle at points A and B. Let C be the vertex on P_1 for which OC bisects AB at D. Let $\theta = \angle BOC$ (see figure).



It suffices to prove that $BD + BC \geq 2\theta$, or equivalently, $\sin\theta + \tan\theta \geq 2\theta$.

Note that $\tan(\theta/2) \geq \theta/2$, so it suffices to prove that $\sin\theta + \tan\theta \geq 4\tan(\theta/2)$. This identity can be verified in a straightforward manner by using the double angle formulas for sine and tangent and the fact that $0 < \theta < \pi/2$.

4. Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k.

Sol. Suppose that $2^s \leq n < 2^{s+1}$. Then the highest power of 2 in $n!$ is $\lfloor n/2 \rfloor + \lfloor n/2^2 \rfloor + \ldots + \lfloor n/2^s \rfloor \leq n/2 + n/4 + \ldots + n/2^s = n(1-1/2^s)$, with equality if and only if $n = 2^s$.

Suppose 2^{n-1} divides $n!$. Then $n - 1 \leq n(1-1/2^s)$, or equivalently, $n/2^s \leq 1$. But $n/2^s \geq 1$ by our choice of s, and therefore $n = 2^s = 2^{k-1}$ where $k = s + 1$ is a positive integer.

Suppose $n = 2^{k-1}$ for some positive integer k. Set $s = k - 1$. From the first paragraph, the highest power of 2 in $n!$ is $n(1-1/2^s) = n(1-1/n) = n-1$; that is to say, 2^{n-1} divides $n!$.

5. Let $1 < x_1 < 2$, and, for $n = 1,2, \ldots,$ define $x_{n+1} = 1 + x_n - \frac{1}{2}x_n^2$. Prove that, for $n \geq 3$, $|x_n - \sqrt{2}| < 2^{-n}$.

Sol. The function $f(x) = 1 + x - \frac{1}{2}x^2 = \frac{3}{2} - \frac{1}{2}(x-1)^2$ is decreasing on the interval (1,2), and it follows that $1 < x_2 < 3/2$ and $11/8 < x_3 < 3/2$. It is routine to check that $|x_3 - \sqrt{2}| < 1/8$.

We proceed by induction. Suppose that $|x_n - \sqrt{2}| < 1/2^n$, $n \geq 3$. We want to write $x_{n+1} - \sqrt{2}$ in terms of $x_n - \sqrt{2}$, and we find that $\sqrt{2} - x_{n+1} = \frac{1}{2}(x_n - \sqrt{2})^2 + (\sqrt{2}-1)(x_n - \sqrt{2})$. It follows that $|x_{n+1} - \sqrt{2}| \leq \frac{1}{2}2^{-2n} + (\sqrt{2}-1)2^{-n}$ and it is routine to show that this is smaller than 2^{-n-1}. This completes the inductive step.

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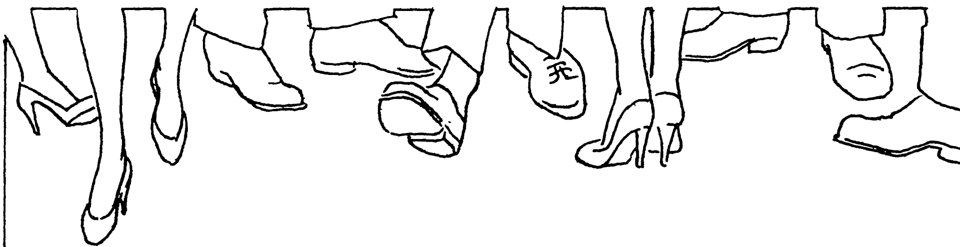

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